# Optimum Sparse Array Beamforming for General Rank Signal Models

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***Abstract*—Sparse array design professes several advantages over their uniform array counterparts, including high resolution and ability to deal with large number of sources in the field of view (FOV). In this paper, we examine sparse arrays achieving maximum signal-to-interference plus noise ratio (MaxSINR) for three different cases, namely, single point source, multiple point sources and Gaussianly spread source, operating in an interference active environment. Our approach does not require any apriori knowledge of the source directions of arrival and their respective power. We formulate the problem as quadratically constraint quadratic program (QCQP), where the cost function is penalized with weighted $l_1$-norm squared of the beamformer weight vector, and propose an iterative technique to control the desired sparsity. It is shown that the optimum sparse array utilizes the array aperture effectively and provides considerable performance improvement over a compact uniform linear array (ULA). Simulation results are presented to show the effectiveness of proposed algorithm for array configurability in the case of both single and general rank signal correlation matrices.**

## I. Introduction

Sparse array design has been a popular approach to alleviate the computational and hardware overhead of the system while optimizing sensor locations to achieve optimality for some predetermined performance criteria. Many different metrics have been proposed for optimal sparse array design depending on the signal processing task at hand. For instance, minimum redundancy criteria and extended aperture coarrays for high resolution direction of arrival (DOA) estimation strive for maximizing the available sensor correlation lags, and efficient structured array topologies. Common examples are minimum redundancy arrays, nested and coprime arrays [1]–[3]. Recently, the enabling switched antenna and beam technologies have motivated the design for environment adaptive sparse arrays. Maximum signal to noise ratio (MaxSNR) and MaxSINR have been shown to yield significantly efficient beamforming with its performance depending largely on the positions of the sensors as well as the locations of sources in the FOV [4]–[7].

Desired source signal power estimation and enhancement in an interference active environment is an important and ubiquitous task in array signal processing. This problem has a direct bearing on improving target detection and localization for radar signal processing, boosting the throughput or channel capacity for MIMO wireless communication systems and improving the resolution capability for medical imaging applications [8]–[10]. Capon method is a well known high resolution data dependent beamforming approach to optimally estimate the desired source signal power by improving the output signal-to-interference plus noise ratio (SINR) [11]. A natural extension of capon beamforming and a broader approach in enhancing desired source power estimation using linear combiner at the receiver amounts to maximizing the SINR over all possible sparse array configurations.

The problem of MaxSINR has been recently investigated in the case of multiple desired point sources [12], [13]. But the nominal approach cannot be extended to spatial spread sources in a straightforward way, as it assumes apriori knowledge of the directions of arrival and power of the source signals which are not typically known [10], [14]. Our proposed technique relaxes this condition and operates directly on the received data correlation matrix. We assume that either the full array data correlation matrix is known or has a co-array that delivers the correlation values across all array elements [15]–[18].

In this paper, we examine MaxSINR sparse arrays for single and higher rank signal correlation matrices. We analyze the performance for the case of single rank correlation matrix which arises when there is one desired source signal in the FOV. We also consider the case of higher rank signal model arises in local scattering and could follow certain statistical or geometric distributions. This is the case of Gaussian or circularly spread sources in wireless communications, due to multipath, or fluctuating source wavefront problem in sonar signal processing [14], [19]–[21]. The multipath can be modeled as statistically independent point sources for the receiver array beamformer. We show the merits of sensor array locations and importance of sparse array configurability in all the above mentioned cases by comparing the sparse array with the commonly used compact ULA.

We pose the problem as optimally selecting $K$ antennas out of $N$ possible equally spaced locations. The antenna selection problem for maximizing SINR amounts to maximizing the principal eigenvalue of the product of correlation matrices [14]. It is an NP hard optimization problem. In order to realize convex relaxation and avoid computational burden of singular value decomposition (SVD) for each possible configuration, we pose this problem as QCQP with weighted $l_1$-norm squared to promote sparsity. We adopt an iteration based approach to

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control the sparsity of the optimum weight vector so that $K$ antenna sensors are finally selected. The weighted $l_1$-norm convex relaxation has been exploited for antenna selection problem for beampattern synthesis, whereas, weighted $l_1$-norm squared relaxation is shown to be very effective for minimizing the required antennas in multicast beamforming [5], [22], [23].

The rest of the paper is organized as follows: In the next section, we state the problem formulation for maximizing the output SINR under general rank signal correlation matrix. Section III deals with the optimum sparse array design by semidefinite relaxation and propose iterative algorithm of finding optimum $K$ antenna sparse array design. Simulation and conclusion follow at the end.

## II. Problem Formulation

Consider $P$ independent desired sources and $Q$ interfering source signals impinging on a linear array with $N$ uniformly placed antennas. Then, the signal received at the array at time instant $t$ is given by:

$$\mathbf{x}(t) = \sum_{k=1}^{P}(\alpha_k(t))\mathbf{s}(\theta_k) + \sum_{l=1}^{Q}(\beta_l(t))\mathbf{i}(\theta_l) + \mathbf{n}(t), \quad (1)$$

where $\mathbf{s}(\theta_k)$ and $\mathbf{i}(\theta_l)$ are the steering vectors corresponding to the respective directions of arrival, $\theta_k$ or $\theta_l$, defined as follows:

$$\mathbf{s}(\theta_k) = [1 \;\; e^{j(2\pi/\lambda)dcos(\theta_k)} \ldots e^{j(2\pi/\lambda)d(N-1)cos(\theta_k)}]^T. \quad (2)$$

The inter-element spacing is given by $d$, $(\alpha_k(t), \beta_l(t)) \in \mathbb{C}$ denote the complex amplitudes of incoming signals, $\mathbf{n}(t) \in \mathbb{C}^N$ represents the additive Gaussian noise with variance $\sigma_n^2$ at the receiver output. The received signal $\mathbf{x}(t)$ is combined linearly by the $N$-antenna beamformer that strives to maximize the output SINR. The output signal $y(t)$ of the optimum beamformer for maximum SINR is given by [14],

$$y(t) = \mathbf{w_0}^H\mathbf{x}(t), \quad (3)$$

where $\mathbf{w_0}$ is the solution of the following optimization problem:

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^H\mathbf{R_{in+n}}\mathbf{w}, \\ \text{s.t.} \quad & \mathbf{w}^H\mathbf{R_s}\mathbf{w} = 1. \end{aligned} \quad (4)$$

For statistically independent signals, $\mathbf{R_s} = \sum_{k=1}^{P}\sigma_k^2\mathbf{s}(\theta_k)\mathbf{s}^H(\theta_k)$ where, $\sigma_k^2 = E\{\alpha_k(t)\alpha_k^H(t)\}$ and $\mathbf{R_{in+n}} = \sum_{l=1}^{Q}(\sigma_l^2\mathbf{i}(\theta_l)\mathbf{i}^H(\theta_l)) + \sigma_n^2\mathbf{I_{N*N}}$ with $\sigma_l^2 = E\{\beta_l(t)\beta_l^H(t)\}$. The problem in (4) can be written equivalently by replacing $\mathbf{R_{in+n}}$ with $\mathbf{R} = \mathbf{R_s} + \mathbf{R_{in+n}}$ as follows [14],

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^H\mathbf{R}\mathbf{w}, \\ \text{s.t.} \quad & \mathbf{w}^H\mathbf{R_s}\mathbf{w} = 1. \end{aligned} \quad (5)$$

The analytical solution of the above optimization problem exists and is given by $\mathbf{w_0} = \mathscr{P}\{\mathbf{R_{in+n}}^{-1}\mathbf{R_s}\} = \mathscr{P}\{\mathbf{R}^{-1}\mathbf{R_s}\}$. The operator $\mathscr{P}\{.\}$ computes the principal eigenvector of it's argument. Substituting $\mathbf{w_0}$ into (3) yields the corresponding optimum output $SINR_o$:

$$SINR_o = \frac{\mathbf{w_0^H}\mathbf{R_s}\mathbf{w_0}}{\mathbf{w_0^H}\mathbf{R_{in+n}}\mathbf{w_0}} = \lambda_{max}\{\mathbf{R_{in+n}^{-1}}\mathbf{R_s}\}. \quad (6)$$

Equation (6) shows that the optimum beamformer for maximizing SINR is directly related to the desired and interference plus noise correlation matrix. It can be shown that for the case of Gaussian spread source, assuming sufficiently high number of independently scattered signals, the source correlation matrix $\mathbf{R_s}$ can well be approximated by [19],

$$\mathbf{R_s} \approx (\mathbf{s}(\theta_0)\mathbf{s}^H(\theta_0)) \circ \mathbf{B}, \quad (7)$$

where '$\circ$' denotes the Hadamard product and $\mathbf{B}$ is given by:

$$\mathbf{B}(m,n) = e^{-2(\pi\delta(m-n))^2\sigma_0^2cos^2(\theta_0)}. \quad (8)$$

The rank of $\mathbf{R_s}$ in (7) is greater than one and its eigenvalue spread depends on the center angle ($\theta_0$) and angle spread ($\sigma_0$). In the following section, we demonstrate array configurability for maximizing SINR for the general rank signal correlation matrix $\mathbf{R_s}$ which could assume unit or higher rank.

## III. Optimum sparse array design

The problem of maximizing the principal eigenvalue of the correlation matrices associated with $K$ antenna selection is a combinatorial optimization problem. We assume that we have an estimate of all the correlation lags of the full array received signal correlation matrix. Then, the problem formulated in (5) can be re-written as follows:

$$\begin{aligned} \underset{\mathbf{w}\in\mathbf{C^N}}{\text{minimize}} \quad & \mathbf{w}^H\mathbf{R}\mathbf{w}, \\ \text{s.t.} \quad & \mathbf{w}^H\mathbf{R_s}\mathbf{w} = 1, \\ & ||\mathbf{w}||_0 = K. \end{aligned} \quad (9)$$

Here, $||.||_0$ determines the cardinality of the weight vector $\mathbf{w}$. The problem expressed in Eq. (9) can then be relaxed to induce the sparsity in optimal weight vector $\mathbf{w}$ without placing a hard constraint on the specific cardinality of $\mathbf{w}$, as follows:

$$\begin{aligned} \underset{\mathbf{w}\in\mathbf{C^N}}{\text{minimize}} \quad & \mathbf{w}^H\mathbf{R}\mathbf{w} + \mu(||\mathbf{w}||_1), \\ \text{s.t.} \quad & \mathbf{w}^H\mathbf{R_s}\mathbf{w} = 1. \end{aligned} \quad (10)$$

Here, $||.||_1$ is the sparsity inducing $l_1$-norm and $\mu$ is a parameter to control the sparsity in the solution. The problem in (10) can be penalized instead by the weighted $l_1$-norm function which is a well known sparsity promoting formulation [24],

$$\begin{aligned} \underset{\mathbf{w}\in\mathbf{C^N}}{\text{minimize}} \quad & \mathbf{w}^H\mathbf{R}\mathbf{w} + \mu(||\mathbf{X^i}\mathbf{w}||_1), \\ \text{s.t.} \quad & \mathbf{w}^H\mathbf{R_s}\mathbf{w} = 1. \end{aligned} \quad (11)$$

where, $\mathbf{X^i}$ is the weight matrix at the $i$th iteration. The weighted $l_1$-norm function in (11) is replaced by the $l_1$-norm squared function without effecting the regularization property of the weighted $l_1$-norm function [5],

$$\begin{aligned} \underset{\mathbf{w}\in\mathbf{C^N}}{\text{minimize}} \quad & \mathbf{w}^H\mathbf{R}\mathbf{w} + \mu(||\mathbf{X^i}\mathbf{w}||_1^2), \\ \text{s.t.} \quad & \mathbf{w}^H\mathbf{R_s}\mathbf{w} = 1. \end{aligned} \quad (12)$$

The SDP relaxation of the above problem can then be realized by replacing $\mathbf{W} = \mathbf{w}\mathbf{w^H}$. Re-expressing the quadratic function, $\mathbf{w}^H\mathbf{R}\mathbf{w} = \text{Tr}(\mathbf{w}^H\mathbf{R}\mathbf{w}) = \text{Tr}(\mathbf{R}\mathbf{w}\mathbf{w}^H) = \text{Tr}(\mathbf{R}\mathbf{W})$,

TABLE I: Proposed algorithm to achieve desired cardinality of optimal weight vector $\mathbf{w_0}$.

| | Steps of proposed algorithm |
|---|---|
| Step 1 | Initialize the weight vector $\mathbf{X^i}$ to all ones and sufficiently small values of $\mu$ and $\epsilon$. |
| Step 2 | Run the rank relaxed SDP of Eq. (13). Check if some entries in $\tilde{\mathbf{W}}$ is exactly zero, if yes, check the cardinality of non zero column of $\tilde{\mathbf{W}}$ and go to Step 1 and increase or decrease the value of $\mu$ to enhance or reduce the sparsity respectively until desired cardinality is achived. If desired cardianlity is achieved go to Step 4 otherwise, in case of no-non zero values go to Step 3. |
| Step 3 | Update the weight vector $\mathbf{X^i}$ according to Eq. (15) and repeat Step 2. |
| Step 4 | After achieving the desired cardinality, run SDP for reduced size correlation matrix corresponding to nonzero values of $\tilde{\mathbf{W}}$ and $\mu = 0$, yielding, $\mathbf{w_0} = \mathscr{P}\{\mathbf{W}\}$. |

where Tr(.) is the trace of the matrix. This expression yields the following problem [5], [25], [26],

$$\begin{aligned} \underset{\mathbf{W}\in\mathbf{C}^{\mathbf{N}*\mathbf{N}},\tilde{\mathbf{W}}\in\mathbf{R}^{\mathbf{N}*\mathbf{N}}}{\text{minimize}} \quad & \text{Tr}(\mathbf{RW}) + \mu\text{Tr}(\mathbf{X^i}\tilde{\mathbf{W}}), \\ \text{s.t.} \quad & \text{Tr}(\mathbf{R_s W}) = 1, \\ & |\mathbf{W}| \geq \tilde{\mathbf{W}}, \\ & \mathbf{W} \succeq 0, \, \text{Rank}(\mathbf{W}) = 1. \end{aligned} \tag{13}$$

The rank constraint in Eq. (13) is non convex. The rank relaxed approximation works well for the underlying problem. Alternatively, one could minimize the nuclear norm of $\mathbf{W}$, as a surrogate for $l_1$-norm in the case of matrices, to induce sparsity in the eigenvalues of $\mathbf{W}$ and promote rank one solutions [27], [28]. The resulting rank relaxed SDP is given by:

$$\begin{aligned} \underset{\mathbf{W}\in\mathbf{C}^{\mathbf{N}*\mathbf{N}},\tilde{\mathbf{W}}\in\mathbf{R}^{\mathbf{N}*\mathbf{N}}}{\text{minimize}} \quad & \text{Tr}(\mathbf{RW}) + \mu\text{Tr}(\mathbf{X^i}\tilde{\mathbf{W}}), \\ \text{s.t.} \quad & \text{Tr}(\mathbf{R_s W}) = 1, \\ & |\mathbf{W}| \geq \tilde{\mathbf{W}}, \\ & \mathbf{W} \succeq 0. \end{aligned} \tag{14}$$

As suggested in [24], the weight matrix $\mathbf{X^i}$ is initialized unweighted i.e. by all ones matrix and iteratively updated as follows,

$$\mathbf{X}^{i+1}_{m,n} = \frac{1}{|\mathbf{W}^{i}_{m,n}| + \epsilon}. \tag{15}$$

The proposed algorithm for controlling the sparsity of the optimal weight vector $\mathbf{w_0}$ is summarized in the TABLE. I.

## IV. Simulations

In this section, we show the effectiveness of our proposed technique for the sparse array design for MaxSINR. The importance of array configurability for MaxSINR is further emphasized and reinforced by comparing the optimum sparse array design with compact ULA performance, under different source signal models. For all our examples, we select $K = 8$ sensors from $N = 16$ possible equally spaced locations with inter-element spacing of $\lambda/2$.

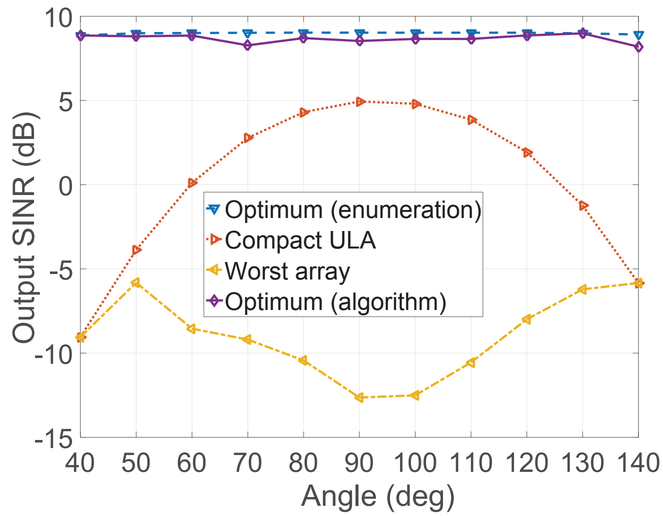
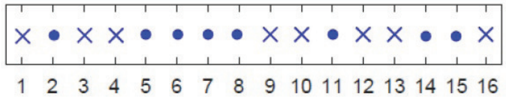


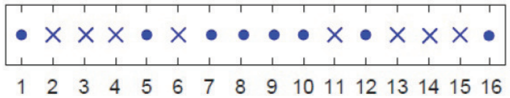

Fig. 1: Output SINR for different array topologies

### *A. Single point source*

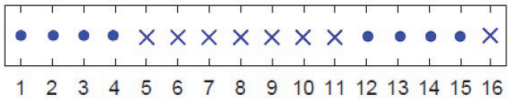
Figure 1 shows the output SINR for different array configurations for the case of single desired point source with its DOA varying from $40^0$ to $140^0$. Three strong interferers are located at $20^0$ and $\pm10^0$ degree apart from the desired source angle. For instance, when the source is at $60^0$, we consider that directions of arrival of interferers are at $40^0$, $50^0$ and $70^0$. The SNR of the desired signal is 0dB, and the INR of each interfering signals is set to 20dB. The theoretical maximum SINR possible is 9.03dB, which corresponds to the array gain offered by 8 antenna array. The case of this MaxSINR arises when interferers are significantly suppressed in the array output. It can clearly be seen from the Fig. 1 that the proposed algorithm performs very close to the optimum array found by exhaustive search (12870 possible configurations), which has very high computational cost attributed to expensive singular value decomposition (SVD) for each enumeration. For the relaxed SDP, we initialize small values for $\mu$ and $\epsilon$ ($10^{-3}$ in our case). On average, the proposed algorithm takes six to seven iterations to converge at the optimum locations and number of sensors; hence, offering dramatic saving in the computational cost. It is of interest to compare the optimum sparse array performance with the compact ULA. It can be seen from Fig. 1, that the optimum sparse array offers considerable SINR advantage over the compact ULA for all the source desired angles of arrival. The performance of the compact ULA degrades severely when the source of interest is more towards the array end-fire location, as 8 antenna element compact ULA fails to resolve and cancel the strong interferers while maintaining unit gain towards the source of interest. For the case of the desired source at the array broadside, the optimum array design found through the proposed algorithm yields an output SINR of 8.6dB, which is 0.3dB less than the corresponding SINR of the optimum array found through exhaustive search. The arrays obtained for the source at broadside are shown in the Fig. 2 (where "." and "×"




(a)

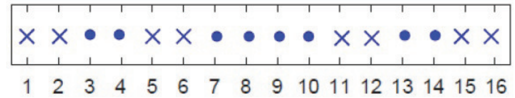



(b)

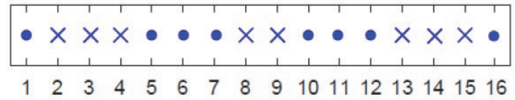



(c)

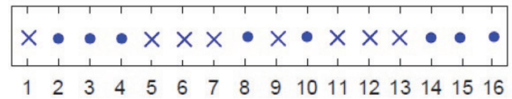
Fig. 2: Array configurations obtained for the point source at the array broadside (a) Optimum (enumeration) (b) Optimum (algorithm) (c) Worst array configuration




(a)




(b)




(c)

Fig. 4: (a) Optimum $8$ antenna array (spread source) (b) Optimum $8$ antenna array (point source) (c) Optimum $8$ antenna array for multiple desired sources (proposed algorithm)

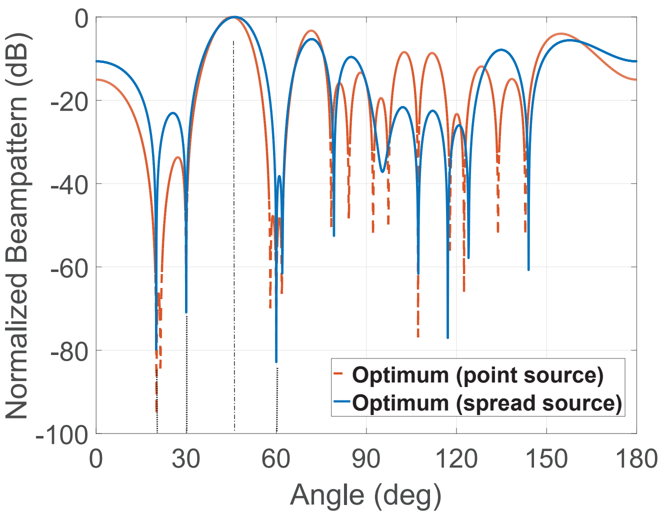



Fig. 3: Beampattern for Gaussian spread source at $45^0$

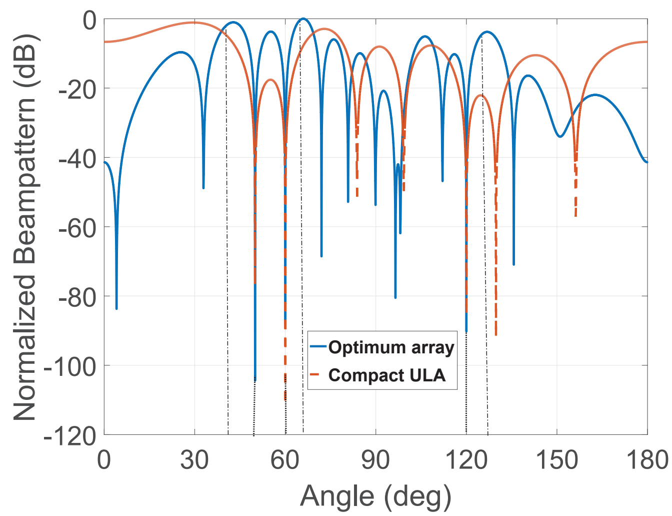



Fig. 5: Beampattern for multiple point sources

represent the presence and absence of sensor respectively). It is worth mentioning that although the worst sparse array configuration utilizes maximum array aperture (Fig. 2c), yet it manages output SINR as low as $-13$dB. This emphasizes the fact that if an arbitrary sparse array structure is employed, it could degrade the performance catastrophically and perform far worst than the compact ULA, which offers modest output SINR of $5$dB for the underlying scenario.

### *B. Gaussian spread source*

Consider the case of a Gaussianly spread source with the center angle of $45^0$ and three interfering signals at $20^0$, $30^0$ and $60^0$. The SNR of this source is $0$dB with power uniformly distributed among all scatterers, and spatial spread of $5^0$. The SNR of interferences is set at $20$dB each. Fig. 3 compares the beampattern of the optimum sparse array in the case of spread source with that of the point source. The optimum sparse array which maximizes the SINR for the Gaussian spatial spread, is shown in the Fig. 4a, and the optimum sparse array for MaxSINR for the point source is shown in the Fig. 4b. Our proposed algorithm, in both cases, recovers the optimum sparse array as that obtained through enumeration. The optimum sparse array for the spread source gives output SINR of $7.5$dB which is $6$dB more as compared with compact ULA and is $0.6$dB more than what is offered by the optimum sparse array for point source. From Fig. 3, we observe that for spread source, the main-lobe is widened to better capture the spread signal power. This beam flatttening feature of Gaussian spread is useful trait for robust adaptive beamforming, and the Gaussian taper is usually applied to the correlation matrix to account for the uncertainity in the desired source correlation matrix [10], [14], [29].

### *C. Multiple point sources*

In the case of multiple point sources, three desired signals are impinging from DOAs $40^0$, $65^0$ and $125^0$ with SNR of $0$dB each. Similarly, three strong interferers with SNR of $30$dB each, are active at DOAs $50^0$, $60^0$ and $120^0$. The compact

ULA again performs with 6dB less output SINR as compared to the optimum sparse array (Fig. 4c) obtained through the proposed methodology. This improved performance is evident from the beampattern of both arrays shown in Fig. 5. The optimum sparse array engages all its degrees of freedom to null the interference while maintaining maximum gain towards all sources of interest. Moreover, the beampattern of the optimum array is more desirable than the comacpt ULA as it has lower side-lobe level.

## V. Conclusion

This paper considered optimum sparse array configuration for maximizing the beamformer output SINR for general rank desired signal correlation matrices. It was shown that the weighted $l_1$-norm squared sparsity promoting penality function with iterative sparsity control algorithm is particularly effective in finding the optimum sparse array design with low computational complexity. We showed the effectiveness of our approach for single and multiple desired point sources and spatially spread source correlation matrix. The MaxS-INR optimum sparse array yielded considerable performance improvement over compact ULA in all of the three cases discussed. In all cases, we solved the optimization problem by both the proposed algorithm and enumeration and showed strong agreement between the two methods.

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