Sparsity-Based Direction Finding of Coherent and Uncorrelated Targets using Active Nonuniform Arrays

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Abstract

In this letter, direction-of-arrival (DOA) estimation of a mixture of coherent and uncorrelated targets is performed using sparse reconstruction and active nonuniform arrays. The data measurements from multiple transmit and receive elements can be considered as observations from the sum coarray corresponding to the physical transmit/receive arrays. The vectorized covariance matrix of the sum coarray observations emulates the received data at a virtual array whose elements are given by the difference coarray of the sum coarray (DCSC). Sparse reconstruction is used to fully exploit the significantly enhanced degrees-of-freedom offered by the DCSC for DOA estimation. Simulated data from multiple-input multiple-output minimum redundancy arrays and transmit/receive co-prime arrays are used for performance evaluation of the proposed sparsity-based active sensing approach.

Index Terms — Active sensing, direction finding, sparse reconstruction, coherent targets

I. INTRODUCTION

Direction-of-arrival (DOA) estimation is an important application of array signal processing and is an area of continued research interest [1-4]. The problem of DOA estimation becomes challenging in the presence of coherent sources or a mixture of coherent and uncorrelated sources, which often arise in the presence of multipath propagation. Traditional subspace-based

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DOA estimation techniques, such as MUSIC [5], can no longer be directly applied due to the rank deficiency of the noise-free covariance matrix. Spatial smoothing can be used to restore the rank of the covariance matrix [6]. However, it can only be applied to specific array structures and always results in reducing the degrees-of-freedom (DOF) that are available for DOA estimation.

Sparse reconstruction techniques have also been applied for DOA estimation of coherent sources [7–9]. In [7], an ℓ_1- SVD method is proposed to perform sparsity-based DOA estimation. In this method, the singular value decomposition (SVD) is employed to reduce the dimensionality of the signal model, followed by a mixed $\ell_{1,2}-$ norm minimization, which assumes group sparsity across the time snapshots. The number of resolvable sources in ℓ_1-SVD is limited by the number of sensors in the array. Joint ℓ_0 approximation, which is a related method to ℓ_1-SVD, has been proposed in [8]. This method uses a mixed $\ell_{0,2}-$norm minimization, instead of $\ell_{1,2}$, in order to enforce sparsity in the reconstructed DOAs. Another sparsity-based method for DOA estimation of more correlated sources than sensors was presented in [9]. This method adopts a dynamic array configuration, wherein different sets of elements of a uniform linear array (ULA) are activated in different time slots, and uses sparse reconstruction to estimate the vectorized form of the source covariance matrix to resolve the sources.

All of the aforementioned schemes employ passive or receive-only arrays for DOA estimation. An active or transmit/receive sensing method was proposed in [10] for direction finding in a coherent environment. This method generalizes the spatial smoothing decorrelation technique to encompass active arrays, where the transmitters illuminate the field of view, and the receivers detect the reflections from the targets. The recorded data emulates measurements at the corresponding sum coarray. Using the coarray equivalence principle, the sum coarray measurements can be considered as originating from a virtual transmit/receive array, which, compared to the physical transmit/receive array, provides a different tradeoff between the number of resolvable targets and the maximum number of mutually coherent targets that can be resolved. The number of resolvable targets for this active sensing scheme is limited by the number of receivers in the virtual transmit/receive array. In [11], a sparse reconstruction scheme for DOA estimation in co-located multiple-input multiple-output (MIMO) radar was proposed. The received data is arranged in a vector which emulates measurements at the sum coarray, and either $\ell_1 -$SVD or a reweighted minimization is applied to reconstruct the signal. For this method, the number of resolvable targets is limited by the number of sum coarray elements.

In this letter, we perform DOA estimation of a mixture of coherent and uncorrelated targets by using the covariance matrix of the data vector that emulates measurements at the sum coarray of active nonuniform arrays. In so doing, the number of DOFs is significantly increased, owing to the fact that the vectorized covariance matrix of the sum coarray observations can be thought of as a single measurement at a virtual array whose elements are given by the difference coarray of the sum coarray (DCSC). The DCSC has a much higher number of elements compared to the sum coarray itself [12]. Sparse reconstruction is employed to fully exploit the enhanced DOFs by estimating the vectorized form of the source covariance matrix, which is linearly related to the vectorized data covariance matrix of the sum coarray observations. Two different nonuniform array geometries are considered for performance evaluation using simulated data. The first configuration is the MIMO minimum redundancy array (MRA), which maximizes the number of elements in the DCSC [12], whereas the second is the transmit/receive co-prime arrays [13, 14]. Simulation results clearly demonstrate the superior performance of the proposed scheme over existing methods in terms of the number of resolvable targets for a given number of transmitters/receivers.

The remainder of the letter is organized as follows. In Section II, the signal model for active sensing is reviewed, and the proposed sparsity-based DOA estimation approach is presented. The MIMO MRA and co-prime configurations are also discussed in this section. The performance of the proposed method is evaluated in Section III through numerical simulation, and Section IV concludes the letter.

II. PROPOSED DOA ESTIMATION APPROACH

A. *Signal Model*

We consider an M-element linear transmit array and an N-element linear receive array. The two arrays may or may not share common elements. These arrays are assumed to be co-located so that a target in the far-field appears to have the same direction at all transmitters and receivers. The scene is illuminated by multiple sequential narrowband transmissions of center frequency f_0 from the different transmitters. This group of transmissions, one from each transmitter, is referred to as a single "snapshot". We assume the field of view to consist of Q point targets in directions $[\theta_1, \theta_2, \dots, \theta_Q]$, where θ is the angle relative to broadside of the transmit or receive array. The target distribution consists of both uncorrelated and coherent targets. Then, the output of the receive array can be expressed as an $MN \times 1$ vector [15, 16]

$$\boldsymbol{x}(t) = \sum_{q=1}^{Q} \boldsymbol{a}_t(\theta_q) \otimes \boldsymbol{a}_r(\theta_q) s_q(t) + \boldsymbol{n}(t), \tag{1}$$

where the operator $\otimes$ denotes the Kronecker product, $s_q(t)$ is the reflection coefficient of the qth target at snapshot t, and $\boldsymbol{a}_t(\theta_q)$ and $\boldsymbol{a}_r(\theta_q)$ are the steering vectors of the transmit and receive arrays corresponding to the direction of the qth target, respectively. The mth element of $\boldsymbol{a}_t(\theta_q)$ is given by $\exp(-jk_0 t_m \sin\theta_q)$ where t_m is the location of the mth transmitter and $k_0 = 2\pi f_0/c$ is the wavenumber at frequency f_0 with c being the speed of light, and the nth element of $\boldsymbol{a}_r(\theta_q)$ is given by $\exp(-jk_0 r_n \sin\theta_q)$ where r_n is the location of the nth receiver. The vector $\boldsymbol{n}(t)$ in (1) is the $MN \times 1$ noise vector. The noise is assumed to be independent and identically distributed following a complex Gaussian distribution.

The term $\boldsymbol{a}_t(\theta_q) \otimes \boldsymbol{a}_r(\theta_q)$ in (1) is equivalent to the steering vector of a virtual receive-only array, whose elements are given by the sum coarray of the transmit and receive arrays. The sum coarray elements are defined as the set $\{(r_n + t_m), 0 \le n \le N-1, 0 \le m \le M-1\}$ [17]. Let L be the number of unique elements in the sum coarray. Then, a new $L \times 1$ received data vector can be formed from (1) as

$$\boldsymbol{x}_{sum}(t) = \boldsymbol{A}_{sum}\boldsymbol{s}(t) + \boldsymbol{n}_{sum}(t), \tag{2}$$

where $\boldsymbol{A}_{sum} = [\boldsymbol{a}_{sum}(\theta_1), \boldsymbol{a}_{sum}(\theta_2), \dots, \boldsymbol{a}_{sum}(\theta_Q)]$ is the $L \times Q$ array manifold corresponding to the sum coarray, $\boldsymbol{s}(t) = [s_1(t), s_2(t), \dots, s_Q(t)]^T$, and $\boldsymbol{a}_{sum}(\theta_q)$ is the steering vector of the sum coarray in direction θ_q. It should be noted that if two or more transmit/receive element pairs contribute to the same sum coarray point, either the average or one of the corresponding measurements could be used in $\boldsymbol{x}_{sum}(t)$. The $\ell_1 -$SVD method can be applied to the sum coarray data vector $\boldsymbol{x}_{sum}(t)$ for sparsity-based DOA estimation [11]. However, the maximum number of resolvable targets in this case is limited to the number of unique elements in the sum coarray [18].

B. Correlation Matrix Based Sparse Reconstruction Approach

The $L \times L$ covariance matrix of the sum coarray data can be expressed as

$$\boldsymbol{R}_{sum} = E\{\boldsymbol{x}_{sum}(t)\boldsymbol{x}_{sum}^H(t)\} = \boldsymbol{A}_{sum}\boldsymbol{R}_{ss}\boldsymbol{A}_{sum}^H + \sigma_n^2\boldsymbol{I}, \tag{3}$$

where $E\{\cdot\}$ is the expectation operator, σ_n^2 is the noise variance, and $\boldsymbol{I}$ is an $L \times L$ identity matrix. $\boldsymbol{R}_{ss}$ is the $Q \times Q$ source correlation matrix, which contains the powers of the reflections from the targets on its main diagonal and the cross-correlations between the targets in the off-diagonal terms. In practice, the covariance matrix is estimated by a sample average over multiple snapshots.

In order to perform DOA estimation of the coherent and uncorrelated targets, we estimate $\boldsymbol{R}_{ss}$ using $\boldsymbol{R}_{sum}$ [9]. To this end, we proceed as follows. The angular region of interest is discretized into a finite set of $K \gg Q$ grid points, $\{\theta_{g_1}, \theta_{g_2}, \dots, \theta_{g_K}\}$, with θ_{g_1} and θ_{g_K} being the limits of the search space. The targets are assumed to be located on the grid. Several methods can be used to modify the model in order to deal with off-grid targets [7, 19]. We define the $L \times K$ array manifold $\widetilde{\boldsymbol{A}}_{sum}$ whose columns are the steering vectors corresponding to the defined angles in the grid, and the $K \times K$ $\widetilde{\boldsymbol{R}}_{ss}$ which holds the auto- and cross-correlation between the potential targets at the defined angles. Equation (3) can then be rewritten as

$$\boldsymbol{R}_{sum} = \widetilde{\boldsymbol{A}}_{sum}\widetilde{\boldsymbol{R}}_{ss}\widetilde{\boldsymbol{A}}_{sum}^{H} + \sigma_n^2\boldsymbol{I}. \tag{4}$$

Since $K \gg Q$, $\widetilde{\boldsymbol{R}}_{ss}$ is a sparse matrix. Sparse reconstruction can then be applied to estimate $\widetilde{\boldsymbol{R}}_{ss}$, and consequently resolve the targets. The nonzero terms on the main diagonal of $\widetilde{\boldsymbol{R}}_{ss}$ correspond to the powers of the target reflections present in the field of view, and the nonzero off-diagonal terms correspond to the correlations between the coherent targets. As a result, the target directions can be obtained by identifying the nonzero terms on the main diagonal.

The covariance matrix $\boldsymbol{R}_{sum}$ is vectorized by stacking its columns to form a tall vector, which emulates a single snapshot at a virtual array whose elements are given by the DCSC of the transmit and receive arrays. With the sum coarray containing L unique elements at positions $x_\ell, \ell = 0, \dots, L-1$, the DCSC elements are given by the set $\Omega = \{x_{\ell_1} - x_{\ell_2}, \ell_1 = 0, \dots, L - 1 \text{ and } \ell_2 = 0, \dots, L-1\}$. It can be readily shown that the $L^2 \times 1$ vectorized form of $\boldsymbol{R}_{sum}$ can be expressed as [9, 20],

$$vec(\boldsymbol{R}_{sum}) = \left(\widetilde{\boldsymbol{A}}_{sum}^{*} \otimes \widetilde{\boldsymbol{A}}_{sum}\right) vec\left(\widetilde{\boldsymbol{R}}_{ss}\right), \tag{5}$$

where $vec(\cdot)$ denotes the vectorization operation and the superscript '*' denotes complex conjugation. Given the model in (5), the constrained optimization problem for reconstructing the $K^2 \times 1$ $vec(\widetilde{\boldsymbol{R}}_{ss})$ can be expressed as [21],

$$\widehat{\boldsymbol{R}}_{ss} = \arg\min_{\widetilde{\boldsymbol{R}}_{ss}} \left\| vec(\boldsymbol{R}_{sum}) - \left(\widetilde{\boldsymbol{A}}_{sum}^{*} \otimes \widetilde{\boldsymbol{A}}_{sum}\right) vec\left(\widetilde{\boldsymbol{R}}_{ss}\right) \right\|_2 + \lambda \left\| vec\left(\widetilde{\boldsymbol{R}}_{ss}\right) \right\|_1, \tag{6}$$

where the ℓ_2-norm is the least squares cost function to maintain data fidelity, and the ℓ_1-norm encourages sparsity in the reconstructed vector. The regularization parameter λ is used to control the weight of the sparsity constraint in the overall cost function.

C. Maximum Number of Resolvable Targets

The maximum number of resolvable targets using the proposed method depends on the number of unique lags in the DCSC and the number of coherent targets. Each pair of coherent targets corresponds to two nonzero off-diagonal terms in $\widetilde{\boldsymbol{R}}_{ss}$, and each target contributes a nonzero term on the main diagonal. Due to conjugate symmetry in $\widetilde{\boldsymbol{R}}_{ss}$, only the lower triangle matrix can be estimated. This implies that, instead of K^2 terms, only $K(K+1)/2$ elements of $\widetilde{\boldsymbol{R}}_{ss}$ need to be estimated. According to [22], the sparsity based minimization problem in (6) is guaranteed to have a unique solution under the condition $P \geq 2D$, where P is equal to the number of independent observations or the number of unique elements in the DCSC and D is the number of nonzero terms in the lower triangle of $\widetilde{\boldsymbol{R}}_{ss}$, which can be expressed as $D = Q + C$, where C is the number of pairs of coherent targets.

The number of unique lags P in the DCSC is a function of the transmit and receive array geometries. For a given number of transmitters and receivers, active array configurations specifically designed to be optimal in the sense that the number of unique elements in the DCSC is maximized, would yield the highest number of resolvable sources. MIMO MRAs are one such type of arrays which are designed under the constraint that the DCSC has no holes [12]. However, the use of such optimal array configurations is not mandatory, and the proposed technique can be applied to other nonuniform arrays, such as co-prime arrays. Co-prime arrays consist of two interleaved ULAs with co-prime number of elements and co-prime element spacing [13, 14]. Table I summarizes the number of unique elements in the sum coarray and the DCSC of three different implementations (Configurations A, B, and C) of a co-prime array comprising a $(2M_c - 1)$ element ULA with $N_c\lambda_0/2$ inter-element spacing and a second ULA having N_c elements spaced by $M_c\lambda_0/2$; M_c and N_c are co-prime integers, the two ULAs share the first element at 0, and λ_0 is the wavelength at the frequency f_0. Configuration A uses the first ULA to transmit and the second ULA to receive. Configuration B employs the first ULA for transmission and both ULAs for reception. Configuration C uses the entire co-prime array to transmit and receive. These implementations provide different tradeoffs between cost, hardware complexity, and the maximum number of unique elements in the DCSC. We observe from Table I that the advantage of the proposed method over the $\ell_1 -$ SVD method applied directly to the sum coarray of the co-prime arrays is more evident for higher values of M_c and N_c. For large M_c and N_c values, a three-fold increase in the DOFs occurs for configurations B and C.

III. Numerical Results

In this section, DOA estimation results for the proposed sparse reconstruction technique using nonuniform active arrays are presented, and a comparison with the $\ell_1 -$ SVD method is also provided. Both MIMO MRAs and co-prime arrays are considered. The root mean square error (RMSE) with respect to the directions is used to compare the two methods.

In the first example, we consider a MIMO MRA, which consists of two receivers positioned at $[0, 7d_0]$ and three transmitters positioned at $[0, d_0, 3d_0]$, where $d_0 = \lambda_0/2$. Fig. 1 shows the corresponding sum coarray and the DCSC. The sum coarray consists of six elements positioned at $[0, 1, 3, 7, 8, 10]d_0$, whereas the DCSC consists of 21 consecutive virtual elements and its aperture extends from $-10d_0$ to $10d_0$. As such, $\ell_1 -$ SVD applied to the sum coarray measurements can estimate up to six sources, whereas the proposed method can estimate up to ten nonzero elements in the lower triangle of the source covariance matrix. This is tested by first considering six targets at directions $[-60°, -20°, -15°, 10°, 30°, 40°]$, with the reflections from the first three targets being mutually coherent. The total number of snapshots is set to 500. Spatially and temporally white Gaussian noise is added to the observations, and the SNR for the six targets is set to $[10, 0, 5, 0, 10, 0]$ dB. The search space is discretized uniformly from $-90°$ and 90° with 1° increment, and the regularization parameter λ is set empirically to 0.5 for the proposed method. The normalized spectrum obtained using $\ell_1 -$ SVD and averaged across the snapshots is shown in Fig. 2(a). Fig. 2(b) depicts the normalized values on the main diagonal of the estimated source covariance matrix using the proposed approach. The dashed vertical lines in both figures indicate the true target directions. We observe that the proposed method has correctly estimated the target directions. However, $\ell_1 - SVD$ misses two targets with low SNR, and produces biased estimates for the remaining targets. The RMSE is 0° for the proposed method.

Next, the same MIMO MRA is used, but the number of targets is increased to seven with the first three being mutually coherent. The targets are positioned at $[-55°, -40°, -15°, 5°, 20°, 45°, 65°]$. A 10 dB SNR is used for all the targets. The regularization parameter λ is set to 0.3. Figs. 3(a) and 3(b) show the estimated spectra using $\ell_1 -$ SVD and the proposed method, respectively. Clearly, $\ell_1 -$ SVD fails to estimate the targets since the total number of targets exceeds the number of sum coarray elements. The proposed method, on the other hand, is successful since the number of nonzero elements in the lower triangle is equal to 10. The corresponding RMSE is $0.24°$. The number of targets is then increased to 10, which is equal to the maximum number of nonzero elements in the lower triangle of the covariance matrix that can be estimated using the proposed method. The target directions are uniformly spaced between $-50°$ and $50°$. The reflections from all the targets are assumed to be uncorrelated in this example, and the other simulation parameters are kept the same as before. Fig. 4(a) shows the estimated spectrum using $\ell_1 -$ SVD, which fails to estimate the target directions because the number of targets is larger than the number of sum coarray elements. The estimated spectrum using the proposed approach is shown in Fig. 4(b). As expected, this method correctly estimates all the DOAs, and the RMSE is equal to $0.2°$ in this example.

Next, we consider a co-prime array with $M_c = 3$ and $N_c = 4$, i.e., the first ULA consists of five physical sensors with positions $[4, 8, 12, 16, 20]d_0$, and the second ULA consists of four sensors positioned at $[0, 3, 6, 9]d_0$. Configuration B is considered, which implies that the first ULA is used to transmit and both ULAs are used to receive. The corresponding sum coarray consists of 25 elements, and the DCSC consists of 67 elements. We consider 30 targets, uniformly spaced between -0.95 and 0.95 in the reduced angular coordinate $\sin(\theta)$, with three targets being mutually coherent. The rest of the simulation parameters are the same as in the previous examples. Figs. 5(a) and 5(b) show the estimated spectra using $\ell_1 -$ SVD and the proposed method, respectively. We observe that $\ell_1 -$ SVD fails to estimate the target directions, since the number of targets exceeds the number of sum coarray elements. The proposed method correctly estimates the DOAs since the number of nonzero elements in the lower triangle of the source covariance matrix in this case is $D = Q + C = 30 + 3 = 33$, and the number of unique elements in the DCSC is $P = 67$ which is greater than $2D$. The corresponding RMSE is $0.03°$.

IV. CONCLUSION

A sparse reconstruction method has been proposed for DOA estimation using active nonuniform arrays. The proposed approach offers a significant enhancement in the DOFs over the currently employed methods by using the covariance matrix of sum coarray measurements to emulate observations at the difference coarray of the sum coarray. The proposed method was tested using two nonuniform array configurations and was shown to successfully estimate the directions of a mixture of coherent and uncorrelated targets.

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TABLE I
NUMBER OF UNIQUE ELEMENTS IN THE SUM COARRAY AND THE DCSA

	Sum Coarray Unique Elements	DCSA Unique Elements
Configuration A	$(2M_c - 1)N_c$	$(5M_c - 3)N_c - M_c$
Configuration B	$(2M_c - 1)(N_c + 1)$	$(7M_c - 5)N_c + M_c$
Configuration C	$(2M_c)(N_c + 1) - 1$	$(7M_c - 3)N_c + M_c$

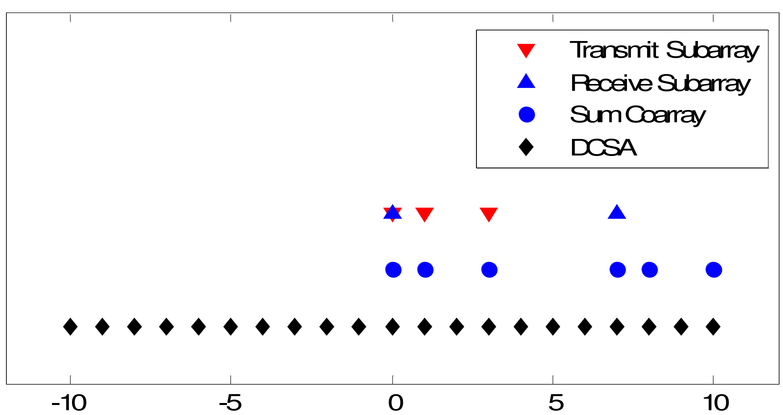
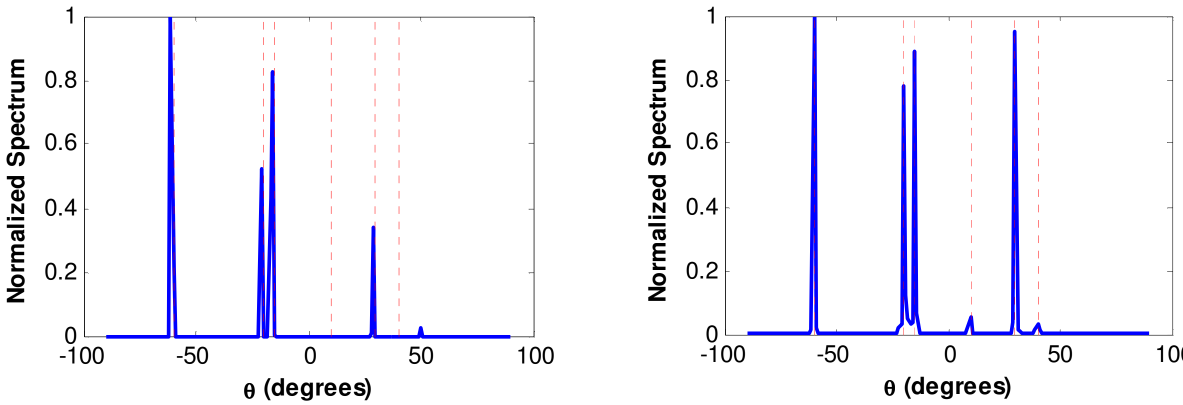
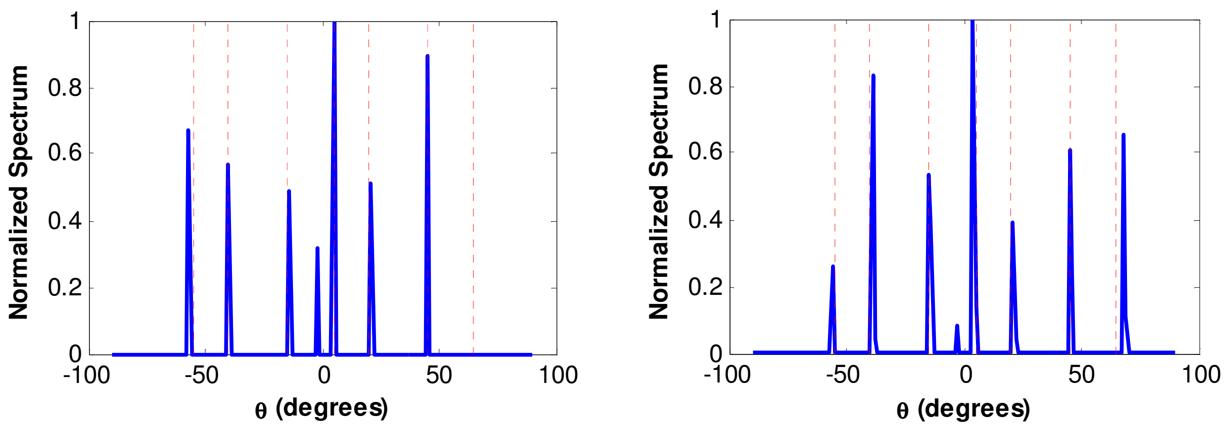


Figure 1. MIMO MRA, sum coarray and DCSA



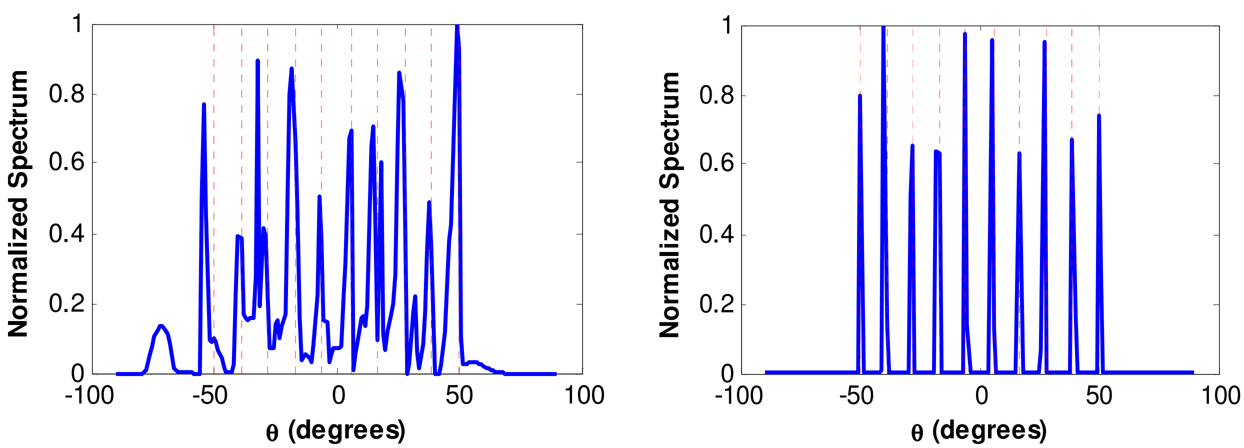
(a) (b)

Figure 2. MIMO MRA, six targets (3 mutually coherent), (a) $\boldsymbol{\ell_1} - \text{SVD}$, (b) Proposed method



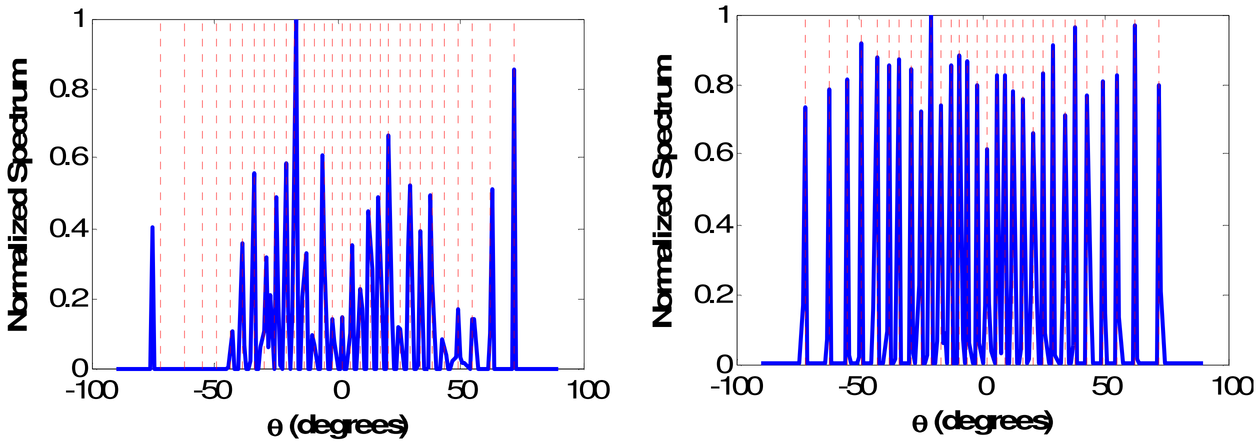
(a) (b)

Figure 3. MIMO MRA, seven targets (3 mutually coherent), (a) $\boldsymbol{\ell_1} - \text{SVD}$, (b) Proposed method



(a) (b)

Figure 4. MIMO MRA, 10 uncorrelated targets, (a) $\boldsymbol{\ell_1} - \text{SVD}$, (b) Proposed method



(a) (b)

Figure 5. Co-prime array, 30 targets (3 coherent), (a) $\boldsymbol{\ell_1} -$ SVD, (b) Proposed method

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