

# Sparsity-Based Extrapolation for Direction-of-Arrival Estimation Using Co-Prime Arrays

Elie BouDaher, Fauzia Ahmad\*, and Moeness G. Amin

Center for Advanced Communications, College of Engineering, Villanova University,  
800 E Lancaster Ave, Villanova, PA 19085, USA

## ABSTRACT

In this paper, we employ a sparsity-based extrapolation technique to extend the usable portion of the difference coarray of a co-prime array for direction-of-arrival (DOA) estimation. The degrees-of-freedom (DOFs) offered by a co-prime array cannot be fully utilized for subspace-based DOA estimation due to the presence of holes in the corresponding difference coarray. The proposed extrapolation approach is first employed to fill the holes in the difference coarray, thereby increasing the available DOFs. MUSIC with spatial smoothing is then applied to the combined set of actual and extrapolated measurements for direction finding. Supporting numerical results are provided that validate the performance enhancements offered by the proposed approach.

**Keywords:** Co-prime arrays, DOA estimation, imputation, sparse reconstruction, extrapolation.

## 1. INTRODUCTION

Direction-of-arrival (DOA) estimation is a major application of array signal processing in a multitude of areas, including radar, sonar, medical imaging, and wireless communications.<sup>1-4</sup> Subspace-based high-resolution DOA estimation techniques, such as MUSIC<sup>5</sup> and ESPRIT<sup>6</sup>, can be applied to an  $N_A$ -element uniform linear array (ULA) to estimate up to  $N_A - 1$  sources. Various sparse or non-uniform array configurations have been proposed which possess the ability to estimate  $O(N_A^2)$  sources using  $N_A$  physical sensors.<sup>7-11</sup> Co-prime arrays constitute one class of non-uniform arrays.<sup>10</sup> A co-prime array consists of two spatially undersampled ULAs with co-prime element spacing and co-prime number of elements. The corresponding set of achievable spatial lags or the difference coarray<sup>12</sup> has missing elements or holes which cause some limitations in DOA estimation.

Three main approaches can be employed to perform DOA estimation using non-uniform arrays in general and co-prime arrays in particular. The first approach uses covariance matrix augmentation.<sup>13-15</sup> However, as the difference coarray of a co-prime array has holes, additional complicated matrix completion processing is required in order for matrix augmentation to fully exploit the offered degrees-of-freedom (DOFs).<sup>15</sup> The second approach vectorizes the covariance matrix to emulate observations at the difference coarray,<sup>11</sup> followed by spatial smoothing.<sup>16</sup> Since the difference coarray of a co-prime array contains holes, the spatial smoothing approach is only applicable to the filled portion of the difference coarray, thereby restricting the DOFs that can be utilized for DOA estimation. In the third approach, sparse signal reconstruction is applied to the vectorized covariance matrix to perform DOA estimation.<sup>17</sup> In this approach, the number of resolvable sources is limited to the number of positive lags in the difference coarray.

Recently, several methods have been proposed to fully exploit the offered DOFs by co-prime arrays.<sup>18-21</sup> In Ref. [18], the authors employed array motion to obtain the measurements corresponding to the missing elements. However, this method requires array displacements and data collection at precise positions. A multi-frequency high-resolution method was proposed in Ref. [19] to fill in the missing elements in the difference coarray and exploit its full aperture. This method imposes restrictions on the sources' power spectra and requires the sources to have a certain bandwidth.

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\* fauzia.ahmad@villanova.edu; <http://www1.villanova.edu/villanova/engineering/research/centers/cac/facilities/rillab.html>

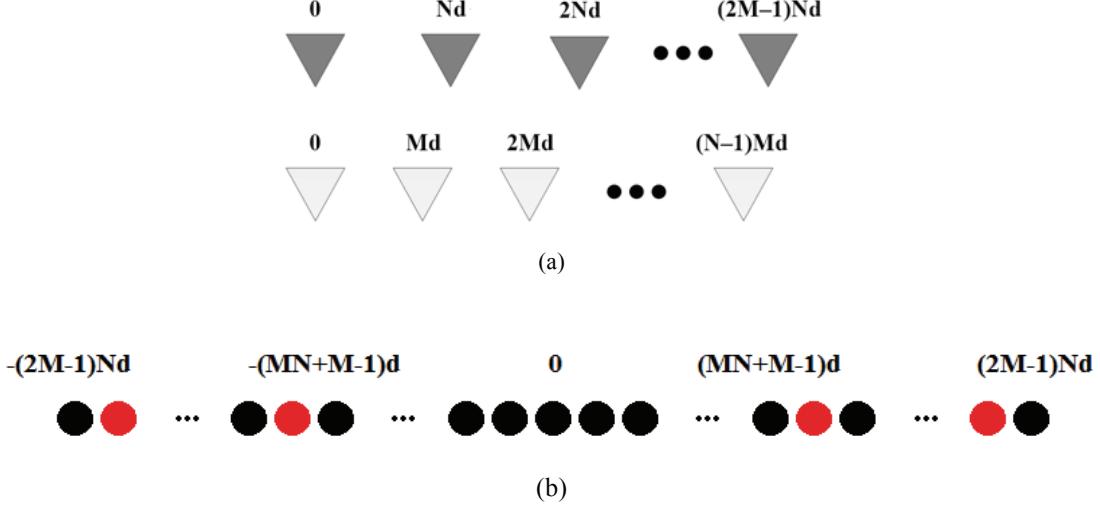


Figure 1. (a) Extended co-prime array. (b) Corresponding difference coarray.

In this paper, we apply a sparsity-based extrapolation technique to fill the holes in the difference coarray. This approach has been extensively used in speech recognition to replace unreliable samples that are heavily corrupted by noise.<sup>22-24</sup> A sparsity-based extrapolation technique was employed in Ref. [25] to extend the aperture of a ULA beyond its physical extent. In this paper, the extrapolation is applied to the vectorized covariance matrix emulating the measurements at the difference coarray. The extrapolated measurements are then combined with the actual measurements to produce the effect of a filled difference coarray with no missing elements. MUSIC with spatial smoothing is then applied to the combined measurement vector. Simulation results validating the improved DOA estimation performance are provided. It is noted that the extrapolation scheme can be used not only to fill the holes but also to extend the difference coarray aperture to beyond that achieved by the physical array.

The remainder of the paper is organized as follows. DOA estimation using co-prime arrays is reviewed in Section 2. In Section 3, the proposed sparsity-based extrapolation technique is presented. Section 4 evaluates the performance of the proposed method, and Section 5 concludes the paper.

## 2. DOA ESTIMATION USING CO-PRIME ARRAYS

### 2.1. Signal Model

We consider an extended co-prime configuration,<sup>11</sup> shown in Fig. 1(a). The first ULA consists of  $2M$  elements with  $Nd$  spacing and the second ULA has  $N$  elements with  $Md$  spacing, where  $M$  and  $N$  are co-prime integers and  $d$  is the unit spacing (often chosen as half-wavelength). Since the two ULAs share an element at position 0, the co-prime configuration consists of  $(2M + N - 1)$  physical sensors. The corresponding difference coarray, shown in Fig. 1(b), extends from  $-(2M - 1)Nd$  to  $(2M - 1)Nd$ , and is filled between  $-(MN + M - 1)d$  and  $(MN + M - 1)d$ .

Assuming that  $K$  narrowband sources with powers  $[\sigma_1^2, \dots, \sigma_K^2]$  impinge on the array from directions  $[\theta_1, \dots, \theta_K]$ , where  $\theta$  is measured relative to broadside, the received data vector can be expressed as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad (1)$$

where  $\mathbf{A}$  is the  $(2M + N - 1) \times K$  array manifold matrix,  $\mathbf{s}(t) = [s_1(t), \dots, s_K(t)]^T$  is the source signal vector at snapshot  $t$ , and  $\mathbf{n}(t)$  is the noise vector of length  $(2M + N - 1)$ . The  $(i, j)$ th element of  $\mathbf{A}$  is given by

$$[\mathbf{A}]_{i,j} = \exp(jkx_i \sin \theta_j), \quad (2)$$

where  $k$  is the wavenumber at the operating frequency,  $x_i$  is the location of the  $i$ th co-prime array element, and  $\theta_j$  is the DOA of the  $j$ th source. Under the assumptions that the sources are uncorrelated and the noise is spatially and temporally white, the covariance matrix of the received measurements can be expressed as

$$\mathbf{R}_{xx} = E\{\mathbf{x}\mathbf{x}^H\} = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma_n^2\mathbf{I}. \quad (3)$$

Here,  $E\{\cdot\}$  is the statistical expectation operator,  $\mathbf{R}_{ss} = \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_K^2)$  is the diagonal source covariance matrix,  $\sigma_n^2$  is the noise variance, and  $\mathbf{I}$  is a  $(2M + N - 1) \times (2M + N - 1)$  identity matrix.

Vectorizing the covariance matrix, we obtain

$$\mathbf{z} = \text{vec}\{\mathbf{R}_{xx}\} = \tilde{\mathbf{A}}\mathbf{p} + \sigma_n^2\tilde{\mathbf{I}} = (\mathbf{A}^* \odot \mathbf{A})\mathbf{p} + \sigma_n^2\tilde{\mathbf{I}} \quad (4)$$

where  $\tilde{\mathbf{A}}$  is the  $(2M + N - 1)^2 \times K$  array manifold matrix corresponding to the difference coarray,  $\mathbf{p} = [\sigma_1^2, \dots, \sigma_K^2]^T$  is the source powers vector,  $\tilde{\mathbf{I}}$  is the vectorized identity matrix, and  $\odot$  denotes the Khatri-Rao product. The  $(2M + N - 1)^2 \times 1$  vector  $\mathbf{z}$  emulates observations at the difference coarray.

In (4), the sources are replaced by their powers and the noise is deterministic. As a result, the model in (4) is similar to that corresponding to a fully coherent environment. Spatial smoothing can be applied to restore the rank of the noise-free covariance matrix of  $\mathbf{z}$  before proceeding with DOA estimation.<sup>11,16</sup> However, due to the restrictions on the array geometries which are required by spatial smoothing, this method can only be applied to the filled part of the difference coarray. A new  $[2(MN + M - 1) + 1] \times 1$  vector  $\mathbf{z}_f$ , which comprises observations at the filled part of the difference coarray, is then formed as

$$\mathbf{z}_f = \tilde{\mathbf{A}}_f\mathbf{p} + \sigma_n^2\tilde{\mathbf{I}}_f, \quad (5)$$

where  $\tilde{\mathbf{A}}_f$  is the  $[2(MN + M - 1) + 1] \times K$  array manifold matrix corresponding to the filled part of the difference coarray and  $\tilde{\mathbf{I}}_f$  is a  $[2(MN + M - 1) + 1] \times 1$  vector whose  $(MN + M)$ th element is equal to one and all remaining elements are zeros. The filled part of the difference coarray is then partitioned into  $(MN + M)$  overlapping subarrays, each having  $(MN + M)$  elements. The received data vector at the  $p$ th subarray ( $p = 1, 2, \dots, MN + M$ ) is denoted by  $\mathbf{z}_{f,p}$  and holds observations at locations determined by the following set

$$\{(m + 1 - p)d, \quad m = 0, 1, \dots, MN + M - 1\}. \quad (6)$$

The overall spatially smoothed covariance matrix is then computed as

$$\mathbf{R}_{zz} = \frac{1}{MN+M} \sum_{p=1}^{MN+M} \mathbf{z}_{f,p} \mathbf{z}_{f,p}^H. \quad (7)$$

MUSIC can then be applied to  $\mathbf{R}_{zz}$ , whose rank is equal to  $(MN + M)$ , to estimate up to  $(MN + M - 1)$  sources.

## 2.2. Sparse Reconstruction

Since MUSIC with spatial smoothing is limited to the filled part of the difference coarray, some of the available DOFs are not exploited. Sparse reconstruction has been used to address this issue and allow the full exploitation of all available DOFs.<sup>17</sup> Using (4), a new vector, comprising the observations at the unique difference coarray elements, can be obtained as

$$\mathbf{z}_u = \tilde{\mathbf{A}}_u\mathbf{p} + \sigma_n^2\tilde{\mathbf{I}}_u. \quad (8)$$

The length of  $\mathbf{z}_u$  is equal to the number of unique elements in the difference coarray, i.e.,  $(3MN + M - N)$ .  $\tilde{\mathbf{A}}_u$  is the  $(3MN + M - N) \times K$  array manifold matrix corresponding to the difference coarray.  $\tilde{\mathbf{I}}_u$  is a  $(3MN + M - N) \times 1$  vector with all zero elements except the  $\frac{(3MN + M - N + 1)}{2}$ th element, which assumes a unit value.

Sparse signal reconstruction can be applied based on the assumption that the sources are sparse in the spatial domain, i.e., only a small number of potential directions are occupied by sources. The angular region of interest is discretized into a set of  $Q$  ( $Q \gg K$ ) grid points,  $\{\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_Q\}$ , with  $\bar{\theta}_1$  and  $\bar{\theta}_Q$  being the limits of the search space. Eq. (8) can be rewritten as

$$\mathbf{z}_u = \bar{\mathbf{A}}_u \bar{\mathbf{p}} + \sigma_n^2 \tilde{\mathbf{I}}_u, \quad (9)$$

where the columns of the  $(3MN + M - N) \times Q$  array manifold matrix  $\bar{\mathbf{A}}_u$  are steering vectors corresponding to the defined angles in the grid.  $\bar{\mathbf{p}}$  is a  $K$ -sparse source power vector of length  $Q$ , with its  $K$  nonzero elements corresponding to the powers of the actual sources. DOA estimation proceeds by solving the following minimization problem

$$[\hat{\mathbf{p}}^T, \hat{\sigma}_n^2]^T = \arg \min_{\bar{\mathbf{p}}, \sigma_n^2} \left[ \frac{1}{2} \|\mathbf{z}_u - \bar{\mathbf{A}}_u \bar{\mathbf{p}} - \sigma_n^2 \tilde{\mathbf{I}}_u\|_2 + \lambda \|\bar{\mathbf{p}}\|_1 \right] \text{ subject to } \bar{\mathbf{p}} \geq \mathbf{0}. \quad (10)$$

The constraint  $\bar{\mathbf{p}} \geq \mathbf{0}$  is added to account for the fact that the source powers always assume positive values. The  $\ell_2$  – norm ensures data fidelity and the  $\ell_1$  – norm encourages sparsity in the reconstructed signal.  $\lambda$  is a regularization parameter that controls the sparsity level of the reconstructed signal. For the sparse reconstruction approach, the number of resolvable sources is limited to the number of nonnegative lags in the difference coarray, i.e.,  $(3MN + M - N)/2$ .

### 3. SPARSITY-BASED EXTRAPOLATION FOR DOA ESTIMATION

In this section, we employ sparse reconstruction to extrapolate observations at the missing elements in the difference coarray. This permits full exploitation of all offered DOFs for DOA estimation using MUSIC with spatial smoothing.

A fully populated version of the difference coarray of Fig. 1(b) would have  $2L + 1$  elements located at  $[-Ld, -(L-1)d, \dots, Ld]$  with  $= (2M - 1)N$ . Let  $\bar{\mathbf{A}}_e$  denote the  $(2L + 1) \times Q$  manifold matrix, whose  $q$ th column is the steering vector of the fully populated difference coarray corresponding to the  $q$ th grid point  $\theta_q$ . Starting with the measurement vector at the original difference coarray  $\mathbf{z}_u$ , sparse reconstruction is first used to estimate the source powers vector  $\hat{\mathbf{p}}$  following (10). An estimate of the observations at the fully populated difference coarray is then obtained as

$$\hat{\mathbf{z}}_e = \bar{\mathbf{A}}_e \hat{\mathbf{p}}. \quad (11)$$

The extrapolated measurements  $\hat{\mathbf{z}}_e$  are then combined with the original measurements  $\mathbf{z}_u$  to form the combined measurements vector  $\mathbf{z}_e$ . If the  $l$ th lag in the fully populated difference coarray is also present in the original difference coarray, the corresponding measurement is obtained from  $\mathbf{z}_u$ . On the other hand, if the lag corresponds to a hole in original difference coarray, the corresponding measurement is taken from  $\hat{\mathbf{z}}_e$ . The following equation summarizes the combination procedure,

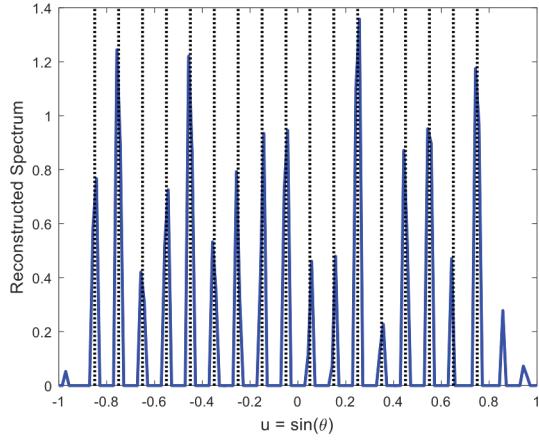
$$\mathbf{z}_e(l) = \begin{cases} \mathbf{z}_u(l), & l \in S \\ \hat{\mathbf{z}}_e(l), & l \notin S \end{cases} \quad (12)$$

where  $\mathbf{z}_e(l)$  denotes the element of  $\mathbf{z}_e$  corresponding to the measurement at lag  $l$ , and  $S$  is the set of element positions of the difference coarray of the co-prime array. Finally, DOA estimation techniques are applied to the combined measurements vector  $\mathbf{z}_e$ . Since the fully populated difference coarray has no missing elements, MUSIC with spatial smoothing is applied to  $\mathbf{z}_e$  in this work.

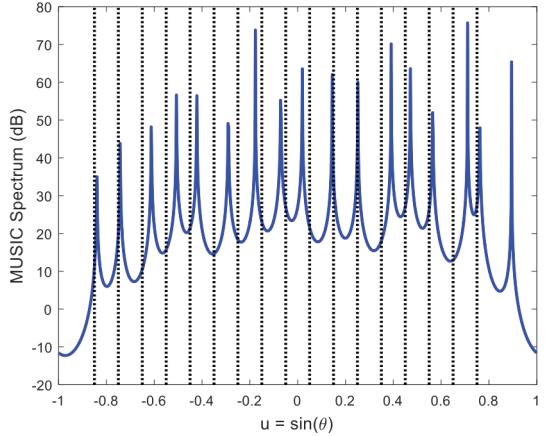
Retaining the available measurements at the original difference coarray is essential for reliable performance of the subsequent DOA estimation. This is because the original measurements contain information about the actual sources, some of which may not be accurately reconstructed or go undetected during the sparse reconstruction step.

### 4. NUMERICAL RESULTS

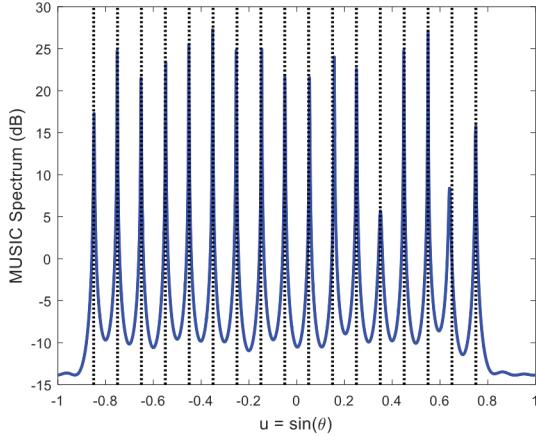
We consider an extended co-prime array with  $M = 3$  and  $N = 5$ . That is, the array comprises 11 elements with positions  $[0, 3, 5, 6, 9, 10, 12, 15, 20, 25]d$ . The corresponding difference coarray extends from  $-25d$  to  $25d$ , and has contiguous elements between  $-17d$  and  $17d$ . A total of 17 sources, uniformly distributed between  $-0.85$  and  $0.75$  in the reduced angular coordinate  $u = \sin(\theta)$  are considered. The SNR for each source is randomly picked from a uniform distribution between  $-5$  dB and  $5$  dB. The total number of snapshots is set to 500. In the sparse reconstruction step, Lasso<sup>26</sup> is applied to minimize the cost function in (10) and obtain an estimate of the source powers vector. The reconstructed spectrum is shown in Fig. 2(a). The directions of the actual sources are shown with vertical dashed lines. It is evident that the reconstructed spectrum contains spurious peaks and one of these peaks is even larger than the power of an actual source. Fig. 2(b) shows the estimated spectrum when MUSIC with spatial smoothing is applied to the measurements at the contiguous part of the difference coarray. Clearly, some of the sources are completely missed and a considerable number of the remaining estimates are biased. Finally, the proposed sparsity-based extrapolation technique is applied to generate measurements at a fully populated difference coarray that extends between  $-25d$  and  $25d$ . In other words, the proposed technique is used to fill in the missing elements in the difference coarray. MUSIC with spatial using is then applied the combined measurement vector and the estimated spectrum is depicted in Fig. 2(c). We observe that all the sources are correctly estimated.



(a)



(b)



(c)

Figure 2.  $M = 3$ ,  $N = 5$ , 17 sources, (a) Estimated spectrum using sparse reconstruction and all measurements from the original difference coarray, (b) MUSIC with spatial smoothing applied to only the filled part of the original difference coarray, (c) MUSIC with spatial smoothing applied to the fully populated difference coarray after extrapolation.

## 5. CONCLUSION

A sparsity-based extrapolation technique was proposed to exploit all of the degrees-of-freedom offered by a co-prime array for DOA estimation. Starting with the observations at the filled part of the difference coarray corresponding to a co-prime configuration, sparse reconstruction was used to extrapolate measurements at the holes in the difference coarray. MUSIC with spatial smoothing was then applied to the combined set of actual and extrapolated measurements. Numerical simulation results were provided that validated the improved DOA performance of the proposed method over the conventional processing of co-prime array measurements.

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