Optimum Adaptive Beamformer Design with Controlled Quiescent Pattern by Antenna Selection

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***Abstract*—In this paper, we consider the design of optimum sparse adaptive beamformer with controlled quiescent beampattern for multiple sources in interference-free environment. The two measures of the maximum output signal-to-noise ratio (SNR) and approximately equal gains towards all sources incident on the array are applied for optimality. We combine both adaptive and deterministic approaches to sparse array design to offer the most general design metric for beamformers. We propose an iterative linear fractional programming method to solve the non-convex antenna selection problem associated with the proposed metric. Simulation examples confirm that the array configuration plays a vital role in determining the beamforming performance in interference-free scenarios.**

***Index Terms*—output SNR, quiescent pattern, linear fractional programming, sidelobe level**

I. Introduction

Adaptive antenna arrays have long been an attractive solution for statistical inference tasks, such as detection and estimation, which arise in many applications, including radar, sonar, communication and satellite navigation [1]–[3]. Antenna arrays are capable of spatial filtering to extract the desired signals while filtering out the unwanted interferences. However, this capability is dependent on the array configuration [4]. With the same number of antennas, different array structures yield different signal to noise ratio (SNR), as well as signal to interference plus noise ratios (SINRs), giving importance to optimum sparse array design. There are many structured sparse arrays wihch have been devised in the literature [5], [6]. These arrays have a primary aim of being able to deal with more sources then physical sensors, but have not been shown to be optimum in SNR or SINR. To achieve optitum array performance, no preset array structure should be enforced, except possibly defining the permissible positions of the sensors in both one and two-dimensional configurations which is equivalent to performing sensor selections.

The effect of unstructured array configurations on interference nulling performance in the case of a single desired source was investigated in our previous work [7], [8]. More recently, the problem of optimum array reconfiguration and antenna selections in the case of multiple desired sources in interference-free environment without applying quiescent pattern constraints was considered in [9]. In this paper, we consider optimum receiver beamformer design with sidelobe constraints in the general case where the dimension of the desired signal subspace is arbitrary and not necessarily confined to a unit value. This case is encountered when multiple communication emitters are present in the field of view, and also occurs in radar tracking of multiple targets. With multiple source signals impinging on the receiver, the optimum weight which provides the maximum output SNR is the principal eigenvector of the source covariance matrix [10]. Clearly, the optimum weight does not guarantee equal gains towards all sources, which may result in performance loss when all sources are identically weak and equally important. Thus, it becomes necessary to investigate the optimum sparse array configuration that maximizes the output SNR while providing equal sensitivities towards all sources.

Sparse array design can be carried out within two frameworks, namely, the deterministic beampattern synthesis and the environment-dependent beamforming. The former focuses on the design of optimum sparse arrays which result in beampatterns with prescribed mainlobe width and reduced sidelobe levels [11], whereas the latter is adaptive and focuses on the sources and noise at hand [12]. Although superior to deterministic design, high sidelobe levels may constitute a potential problem in adaptive beamformer design. This problem becomes more pronounced when directional interferers are suddenly switched on [13]–[15]. In this case, new weights and array structure must be recomputed/reconfigured to respond to the new environment which requires a finite amount of response time. This mandates selections of sensor positions and calculations of associated weights according to desirable constraints on the beamformer sidelobes. As such, an optimization metric combining adaptive beamforming and constraint-based deterministic array design should seek the sparse array configuration that maximizes the output SNR with well-controlled sidelobes. This constitutes the main novelty and focus of this paper. We formulate, analyze, and configure the two optimum adaptive arrays which provide the maximum SNR and equal gains towards all potential incoming sources with controlled sidelobe levels. An iterative linear fractional programming method is proposed to solve the antenna selection problem.

The rest of this paper is organized as follows: We introduce the problem in section II. Formulation of adaptive beamformer design with and without controlled quiescent pattern constraints is elucidated in section III. The problem of deterministic beamformer design is reformulated in section IV,

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circumventing the assumption of symmetric arrays. Simulation results, presented in section V, validate the effectiveness of proposed methods. Finally, concluding remarks are provided in section VI.

II. Problem Formulation

The problem formulation is similar to that of reference [9]. Consider a linear array of K isotropic antennas with positions specified by multiple integer of unit inter-element spacing $x_n d, x_n \in \mathbb{N}, n=1,\cdots,K$. Suppose that p source signals are impinging on the array from directions $\{\theta_1,\cdots,\theta_p\}$ with spatial steering vectors specified by,

$$\mathbf{u}_k=[e^{jk_0x_1d\cos\theta_k},\cdots,e^{jk_0x_Nd\cos\theta_k}]^T, k=1,\cdots,p, \quad (1)$$

respectively. The wavenumber is defined as $k_0=2\pi/\lambda$ with λ being the wavelength and ^{T} denotes transpose operation. The received signal at time instant t is given by,

$$\mathbf{x}(t)=\mathbf{U}\mathbf{s}(t)+\mathbf{n}(t), \quad (2)$$

where $\mathbf{U}=[\mathbf{u}_1,\cdots,\mathbf{u}_p]\in\mathbb{C}^{K\times p}$ is the source array manifold matrix. In the above equation, $\mathbf{s}(t)\in\mathbb{C}^p$ denotes the source vector and $\mathbf{n}(t)\in\mathbb{C}^K$ represents the received noise vector. The output of the K-antenna beamformer is given by,

$$y(t)=\mathbf{w}^H\mathbf{x}(t), \quad (3)$$

where $\mathbf{w}\in\mathbb{C}^K$ is the complex vector of beamformer weights and ^{H} stands for Hermitian operation. With additive Gaussian noise, i.e. $\mathbf{n}(t)\sim\mathcal{CN}(\mathbf{0},\sigma_n^2\mathbf{I})$ with σ_n^2 denoting noise power level, and in the absence of interfering sources, the optimal weight vector for maximizing the output SNR is given by,

$$\mathbf{w}_{\text{opt}}=\mathcal{P}\{\mathbf{R}_s\}=\mathcal{P}\{\mathbf{U}\mathbf{C}_s\mathbf{U}^H\}. \quad (4)$$

where $\mathcal{P}\{\cdot\}$ denotes the principal eigenvector of the matrix, $\mathbf{R}_s$ is defined as $\mathbf{R}_s=\mathbf{U}\mathbf{C}_s\mathbf{U}^H$ with $\mathbf{C}_s=E\{\mathbf{s}(t)\mathbf{s}^H(t)\}$ denoting source auto-correlation matrix. The output SNR is,

$$\text{SNR}_{\text{opt}}=\frac{\mathbf{w}_{\text{opt}}^H\mathbf{R}_s\mathbf{w}_{\text{opt}}}{\mathbf{w}_{\text{opt}}^H\mathbf{R}_n\mathbf{w}_{\text{opt}}}=\frac{\lambda_{\max}\{\mathbf{R}_s\}}{\sigma_n^2}=\frac{\|\mathbf{R}_s\|_2}{\sigma_n^2}. \quad (5)$$

Clearly, array configuration affects output SNR of the optimum beamformer $\mathbf{w}_{\text{opt}}$ through the term of $\|\mathbf{R}_s\|_2$ from Eq. (5) [9].

III. Adaptive Beamformer Design

Suppose that there are N grid points for possible sensor positions and K available antennas for placement. Denote an antenna selection vector $\mathbf{z}\in\{0,1\}^N$ with "zero" entry denoting the corresponding antenna discarded and "one" entry for a selected antenna. As steering vectors are directional, the implementation of antenna selection is clearly expressed as $\mathbf{u}_i(\mathbf{z})=\mathbf{u}_i\odot\mathbf{z}, i=1,\ldots,p$ with $\odot$ denoting element-wise product and, accordingly, $\mathbf{U}(\mathbf{z})=[\mathbf{u}_1(\mathbf{z}),\ldots,\mathbf{u}_p(\mathbf{z})]$. Ideally, the adaptive beamformer should be reconfigured through antenna selection $\mathbf{z}$ such that the spectral norm of the reduced-dimensional source covariance matrix $\|\mathbf{R}_s(\mathbf{z})\|_2$ is maximum. That is,

$$\begin{aligned}\max_{\mathbf{z}} \quad & \|\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\|_2, \\ \text{subject to} \quad & \mathbf{z}\in\{0,1\}^N, \\ & \mathbf{1}^T\mathbf{z}=K.\end{aligned} \quad (6)$$

However, the spectral norm of a matrix is convex and it is a difficult problem to maximize a convex objective function. Moreover, it is computationally prohibitive to implement eigenvalue decomposition for each subset of K sensor locations, even for a small-scaled problem. Thus, we resort to convex relaxation, that is, maximizing the lower bound of the spectral norm of source covariance matrix.

A. Adaptive Beamformer Design Without Sidelobe Constraints

Proceeding from the definition of the spectral norm of the source covariance matrix, we obtain its lower bound [16], i.e.,

$$\begin{aligned}& \|\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\|_2 \\ = \quad & \max_{\|\mathbf{b}\|_2=1} \mathbf{b}^H\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\mathbf{b}, \\ \geq \quad & \tilde{\mathbf{e}}_1^H\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\tilde{\mathbf{e}}_1, \forall\tilde{\mathbf{e}}_1, \|\tilde{\mathbf{e}}_1\|_2=1.\end{aligned} \quad (7)$$

Assume that the source auto-correlation matrix $\mathbf{C}_s$ is known a priori or previously estimated. Then, the principal eigenvector of the array with fully populated grid of N sensors, ($\mathbf{z}=\mathbf{1}$), can be calculated through eigenvalue decomposition of $\mathbf{R}_s$,

$$\mathbf{U}\mathbf{C}_s\mathbf{U}^H=\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H, \quad (8)$$

where $\mathbf{\Lambda}=\text{diag}(\lambda_1,\ldots,\lambda_p,0,\ldots,0)$ with the N eigenvalues populating along its diagonal in a descending order $\lambda_1\geq\ldots\geq\lambda_p$ and $\mathbf{e}_i, i=1,\ldots,N$ are corresponding eigenvectors. The source covariance matrix, upon implementing sensor selection, can be written as,

$$\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})=\mathbf{E}(\mathbf{z})\mathbf{\Lambda}\mathbf{E}^H(\mathbf{z}). \quad (9)$$

Clearly, the set of vectors $\mathbf{E}(\mathbf{z})=[\mathbf{e}_1(\mathbf{z}),\ldots,\mathbf{e}_p(\mathbf{z})]$ constitutes a set of redundant basis of the K-dimensional source subspace, as $\lambda_{p+1}=\ldots=\lambda_N=0$. Furthermore, the vector $\mathbf{e}_1(\mathbf{z})=\mathbf{e}_1\odot\mathbf{z}$ still possesses the largest coefficient λ_1, and, as such, $\mathbf{e}_1(\mathbf{z})/\|\mathbf{e}_1(\mathbf{z})\|_2$ can be utilized to closely approximate the principal eigenvector of the selected sparse array, i.e.,

$$\|\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\|_2\geq\frac{\mathbf{e}_1^H(\mathbf{z})\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\mathbf{e}_1(\mathbf{z})}{\|\mathbf{e}_1(\mathbf{z})\|_2^2}. \quad (10)$$

Therefore, the optimum adaptive beamformer for maximizing the output SNR can be relaxed as [9],

$$\begin{aligned}\max_{\mathbf{z}} \quad & \frac{\mathbf{e}_1^H(\mathbf{z})\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\mathbf{e}_1(\mathbf{z})}{\|\mathbf{e}_1(\mathbf{z})\|_2^2}, \\ \text{subject to} \quad & \mathbf{z}\in\{0,1\}^N,\ \mathbf{1}^T\mathbf{z}=K.\end{aligned} \quad (11)$$

The optimum adaptive beamformer in Eq. (4) provides the maximum output SNR, but cannot guarantee equal sensitivities towards all sources. A large difference in array sensitivities towards the users may not be desirable in practice, specifically when all sources must be equally served by the receiver. If a specific gain toward each source is required, for example equal gains, then the beamformer weights can be set to,

$$\mathbf{w}=\mathbf{U}\beta, \quad (12)$$

where β denotes the desired gain vector towards all sources. Note that the beamformer in Eq. (12) with $\beta=\mathbf{1}$ does not imply exact equal sensitivities due to the spatial correlation among the sources translating into sidelobes. However, if exact gain values are enforced, then we, in essence, consume p degrees of freedom (DoF) with little or no DoF left for SNR considerations. The corresponding sensor selection problem can be formulated as,

$$\begin{aligned} \max_{\mathbf{z}} \quad & \frac{\beta^H \mathbf{U}^H(\mathbf{z})\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta}{\beta^H \mathbf{U}^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta}, \\ \text{subject to} \quad & \mathbf{z} \in \{0,1\}^N, \ \mathbf{1}^T\mathbf{z} = K. \end{aligned} \tag{13}$$

Note that the construction of optimum adaptive beamformers comprises two intertwined steps, sensor management through antenna selection $\mathbf{z}$ and optimum weight calculations, either by Eq. (4) or Eq. (12) according to the underlying applications.

B. Adaptive Beamformer with Controlled Quiescent Pattern

We combine the adaptive and deterministic approaches to offer the most general metric for beamformer design. Specific sidelobe levels can be provided to shape the array response prior to the onset of any directional interference. In this case, the optimization problem becomes more involved due to the additional sidelobe controlling constraints.

Denote the sidelobe angular region as Ω and sample Ω with a set of predefined discrete angles $\{\theta_{s1}, \dots, \theta_{sL}\}$. Their respective steering vectors of the fully populated array are denoted as $\mathbf{a}_i, i = 1, \dots, L$ and $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_L]$. The basic model is to find an optimum sparse array with the maximum output SNR while maintaining low sidelobe levels over the specified region Ω. Denote the required peak sidelobe level (PSL) as γ. We impose the following L quadratic constraints on the sparse array design for maximum output SNR and equal-gain beamformers,

$$\mathbf{e}_1^H(z)\mathbf{a}_i(\mathbf{z})\mathbf{a}_i^H(\mathbf{z})\mathbf{e}_1(z) \leq \gamma \mathbf{e}_1^H(z)\mathbf{e}_1(z), \tag{14}$$

and

$$\beta^H \mathbf{U}^H(\mathbf{z})\mathbf{a}_i(\mathbf{z})\mathbf{a}_i^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta \leq \gamma \beta^H \mathbf{U}^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta, \tag{15}$$

respectively, where $i = 1, \dots, L$. The choice of L is a trade-off between generating well-shaped quiescent pattern and maximizing the output SNR. As the number L is increased for deterministic constraints, the number of DoFs available for adaptive design is reduced.

The adaptive beamformer design for maximizing the output SNR with controlled sidelobe level is expressed as,

$$\begin{aligned} \max_{\mathbf{z}} \quad & \frac{\mathbf{e}_1^H(\mathbf{z})\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\mathbf{e}_1(\mathbf{z})}{\|\mathbf{e}_1(\mathbf{z})\|_2^2}, \\ \text{subject to} \quad & \mathbf{e}_1^H(z)\mathbf{a}_i(\mathbf{z})\mathbf{a}_i^H(\mathbf{z})\mathbf{e}_1(z) \leq \gamma \mathbf{e}_1^H(\mathbf{z})\mathbf{e}_1(\mathbf{z}), \\ & i = 1, \dots, L \\ & \mathbf{z} \in \{0,1\}^N, \ \mathbf{1}^T\mathbf{z} = K. \end{aligned} \tag{16}$$

Define the vectors $\bar{\mathbf{e}}_1 = \mathbf{e}_1^* \odot \mathbf{e}_1$, $\bar{\mathbf{a}}_i = \mathbf{e}_1^* \odot \mathbf{a}_i, i = 1, \dots, L$ and the matrix $\bar{\mathbf{U}} = [\mathbf{e}_1^* \odot \mathbf{u}_1, \dots, \mathbf{e}_1^* \odot \mathbf{u}_p]$ with * denoting the conjugate operation. The problem in Eq. (16) can be rewritten as,

$$\begin{aligned} \max_{\mathbf{z}} \quad & \frac{\mathbf{z}^H \bar{\mathbf{U}}\mathbf{C}_s\bar{\mathbf{U}}^H\mathbf{z}}{\mathbf{z}^H\bar{\mathbf{e}}_1}, \\ \text{subject to} \quad & \mathbf{z}^H\bar{\mathbf{a}}_i\bar{\mathbf{a}}_i^H\mathbf{z} \leq \gamma \mathbf{z}^H\bar{\mathbf{e}}_1, i = 1, \dots, L \\ & 0 \leq \mathbf{z} \leq 1, \ \mathbf{1}^T\mathbf{z} = K. \end{aligned} \tag{17}$$

As every sublevel set $\mathcal{S}_\nu$,

$$\mathcal{S}_\nu = \{\mathbf{z} | \frac{\mathbf{z}^H \bar{\mathbf{U}}\mathbf{C}_s\bar{\mathbf{U}}^H\mathbf{z}}{\mathbf{z}^H\bar{\mathbf{e}}_1} \leq \nu\}, \tag{18}$$

for arbitrary ν, is convex, the objective in Eq. (17) is quasi-convex [17]. We relax the binary constraints $\mathbf{z} \in \{0,1\}^N$ to a box constraint $0 \leq \mathbf{z} \leq 1$, as the global maximizer of a quasi-convex function locates at the extreme points of the polyhedral [18], [19], which, in turn, eliminates the notorious binary constraints. The problem represented by Eq. (17) is a quadratic fractional, which can be transformed into linear fractional programming iteratively. The problem in the $(k+1)$th iteration based on the kth solution $\mathbf{z}^{(k)}$ is written as,

$$\begin{aligned} \max_{\mathbf{z}} \quad & \frac{2\mathbf{z}^{(k)H}\bar{\mathbf{U}}\mathbf{C}_s\bar{\mathbf{U}}^H\mathbf{z} - \mathbf{z}^{(k)H}\bar{\mathbf{U}}\mathbf{C}_s\bar{\mathbf{U}}^H\mathbf{z}^{(k)}}{\mathbf{z}^H\bar{\mathbf{e}}_1}, \\ \text{subject to} \quad & (2\mathbf{z}^{(k)H}\bar{\mathbf{a}}_i\bar{\mathbf{a}}_i^H - \gamma\bar{\mathbf{e}}_1^H)\mathbf{z} - \mathbf{z}^{(k)H}\bar{\mathbf{a}}_i\bar{\mathbf{a}}_i^H\mathbf{z}^{(k)} \leq 0, \\ & i = 1, \dots, L, \\ & 0 \leq \mathbf{z} \leq 1, \ \mathbf{1}^T\mathbf{z} = K. \end{aligned} \tag{19}$$

The linear fractional programming in Eq. (19) can be further transformed into linear programming as follows,

$$\begin{aligned} \max_{\mathbf{y},\alpha} \quad & 2\mathbf{z}^{(k)H}\bar{\mathbf{U}}\mathbf{C}_s\bar{\mathbf{U}}^H\mathbf{y} - \mathbf{z}^{(k)H}\bar{\mathbf{U}}\mathbf{C}_s\bar{\mathbf{U}}^H\mathbf{z}^{(k)}\alpha, \\ \text{subject to} \quad & (2\mathbf{z}^{(k)H}\bar{\mathbf{a}}_i\bar{\mathbf{a}}_i^H - \gamma\bar{\mathbf{e}}_1^H)\mathbf{y} - \mathbf{z}^{(k)H}\bar{\mathbf{a}}_i\bar{\mathbf{a}}_i^H\mathbf{z}^{(k)}\alpha \leq 0, \\ & i = 1, \dots, L, \\ & \mathbf{1}^T\mathbf{y} = K\alpha, \ 0 \leq \mathbf{y} \leq \alpha, \\ & \alpha > 0, \ \bar{\mathbf{e}}_1^H\mathbf{y} = 1. \end{aligned} \tag{20}$$

The optimum selection vector is finally obtained by $\mathbf{z} = \mathbf{y}/\alpha$. Similarly, the design of adaptive beamformer with approximately equal sensitivities towards all sources and controlled sidelobe level is formulated as,

$$\begin{aligned} \max_{\mathbf{z}} \quad & \frac{\beta^H \mathbf{U}^H(\mathbf{z})\mathbf{U}(\mathbf{z})\mathbf{C}_s\mathbf{U}^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta}{\beta^H \mathbf{U}^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta}, \\ \text{subject to} \quad & \beta^H \mathbf{U}^H(\mathbf{z})\mathbf{a}_i(\mathbf{z})\mathbf{a}_i^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta \leq \gamma \beta^H \mathbf{U}^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta, \\ & i = 1, \dots, L, \\ & \mathbf{z} \in \{0,1\}^N, \ \mathbf{1}^T\mathbf{z} = K. \end{aligned} \tag{21}$$

Utilizing the following property of Khatri-Rao product $\circ$,

$$\mathbf{A}\text{diag}(\mathbf{x})\mathbf{b} = (\mathbf{b}^T \circ \mathbf{A})\mathbf{x}, \tag{22}$$

we obtain

$$\mathbf{U}^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta = \mathbf{U}^H\text{diag}(\mathbf{z})\mathbf{U}\beta = [(\mathbf{U}\beta)^T \circ \mathbf{U}^H]\mathbf{z}, \tag{23}$$

and

$$\mathbf{a}_i^H(\mathbf{z})\mathbf{U}(\mathbf{z})\beta = \mathbf{a}_i^H\text{diag}(\mathbf{z})\mathbf{U}\beta = [(\mathbf{U}\beta)^T \circ \mathbf{a}_i^H]\mathbf{z},. \tag{24}$$

Define the vector $\bar{\beta} = (\mathbf{U}\beta) \odot (\mathbf{U}\beta)^*$, the matrices $\bar{\bar{\mathbf{U}}} = (\mathbf{U}\beta)^T \circ \mathbf{U}^H$ and $\bar{\bar{\mathbf{a}}}_i = (\mathbf{U}\beta)^T \circ \mathbf{a}_i^H$. The problem in Eq. (21) can be rewritten as

$$\begin{aligned} \max_{\mathbf{z}} \quad & \frac{\mathbf{z}^T \bar{\bar{\mathbf{U}}}^H \mathbf{C}_s \bar{\bar{\mathbf{U}}} \mathbf{z}}{\mathbf{z}^T \bar{\beta}} \qquad (25) \\ \text{subject to} \quad & \mathbf{z}^T \bar{\bar{\mathbf{a}}}_i \bar{\bar{\mathbf{a}}}_i^H \mathbf{z} \leq \gamma \bar{\beta}^T \mathbf{z}, \\ & \mathbf{0} \leq \mathbf{z} \leq 1, \ \mathbf{1}^T \mathbf{z} = K. \end{aligned}$$

Clearly, the formulation in Eq. (25) also belongs to quadratic fractional programming. The iterative linear fractional programming introduced above can then be utilized to obtain the optimal beamformer with approximately equal sensitivities towards all sources and controlled quiescent pattern.

IV. Deterministic Beamformer Design

The deterministic beamformer design prescribes mainlobe width and reduced sidelobe levels. The N-dimensional weight vector $\mathbf{w}$ for deterministic array design is required to be sparse with cardinality of K. Although this problem has been intensively investigated in the literature [20]–[22], the assumption of symmetric arrays is not required in this work and an effective algorithm is proposed to solve the general design problem. The deterministic array design is formulated as,

$$\begin{aligned} \min_{\mathbf{w}} \quad & \|\mathbf{w}\|_1, \qquad (26) \\ \text{subject to} \quad & \mathbf{w}^H \mathbf{u}_i \mathbf{u}_i^H \mathbf{w} \leq 1+\delta, i=1,\dots,p, \\ & \mathbf{w}^H \mathbf{u}_i \mathbf{u}_i^H \mathbf{w} \geq 1-\delta, i=1,\dots,p, \\ & \mathbf{w}^H \mathbf{a}_i \mathbf{a}_i^H \mathbf{w} \leq \gamma, i=1,\dots,L, \qquad (27) \end{aligned}$$

where δ represents the acceptable mainlobe ripple. We split the $N \times 1$ weight variable into real and imaginary parts and stack them as a $2N \times 1$ vector, i.e., $\tilde{\mathbf{w}} = [\mathbb{R}(\mathbf{w})^T, \mathbb{I}(\mathbf{w})^T]^T$. Then, the problem of deterministic array design can be transformed from the complex domain to the real domain, i.e.,

$$\begin{aligned} \min_{\mathbf{w}} \quad & \|\tilde{\mathbf{w}}\|_{2,1}, \qquad (28a) \\ \text{s.t.} \quad & \tilde{\mathbf{w}}^H \mathbf{U}_i \tilde{\mathbf{w}} \leq 1+\delta, i=1,\dots,p, \qquad (28b) \\ & \tilde{\mathbf{w}}^H \mathbf{U}_i \tilde{\mathbf{w}} \geq 1-\delta, i=1,\dots,p, \qquad (28c) \\ & \tilde{\mathbf{w}}^H \mathbf{A}_i \tilde{\mathbf{w}} \leq \gamma, i=1,\dots,L \qquad (28d) \end{aligned}$$

where the matrices $\mathbf{U}_i$ and $\mathbf{A}_k$ are defined as,

$$\mathbf{U}_i = \begin{bmatrix} \mathbb{R}(\mathbf{u}_i\mathbf{u}_i^H) & -\mathbb{I}(\mathbf{u}_i\mathbf{u}_i^H) \\ \mathbb{I}(\mathbf{u}_i\mathbf{u}_i^H) & \mathbb{R}(\mathbf{u}_i\mathbf{u}_i^H) \end{bmatrix}, \ i=1,\dots,p \qquad (29)$$

and

$$\mathbf{A}_k = \begin{bmatrix} \mathbb{R}(\mathbf{a}_i\mathbf{a}_i^H) & -\mathbb{I}(\mathbf{a}_i\mathbf{a}_i^H) \\ \mathbb{I}(\mathbf{a}_i\mathbf{a}_i^H) & \mathbb{R}(\mathbf{a}_i\mathbf{a}_i^H) \end{bmatrix}, \ k=1,\dots,L \qquad (30)$$

respectively. Moreover, the term $\|\tilde{\mathbf{w}}\|_{2,1}$ is utilized to promote the group sparsity and defined as,

$$\|\tilde{\mathbf{w}}\|_{2,1} = \sum_{i=1}^{N} \sqrt{\mathbb{R}(\mathbf{w}_i)^2 + \mathbb{I}(\mathbf{w}_i)^2}. \qquad (31)$$

Note that the lower bound constraint imposed on the mainlobe in Eq. (28c) is not convex and we approximate it iteratively using its global affine underestimate. Moreover, the group sparsity in Eq. (31) can be achieved through the Second-Order Cone Programming (SOCP) [23]. The $k+1$th iteration of the problem based on the kth solution $\tilde{\mathbf{w}}^{(k)}$ is written as,

$$\begin{aligned} \min_{\mathbf{w},\mathbf{t}} \quad & \mathbf{1}^T \mathbf{t}, \qquad (32) \\ \text{subject to} \quad & 2\tilde{\mathbf{w}}^{kH} \mathbf{U}_i \tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{kH} \mathbf{U}_i \tilde{\mathbf{w}}^k \leq 1+\delta, i=1,\dots,p, \\ & 2\tilde{\mathbf{w}}^{kH} \mathbf{U}_i \tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{kH} \mathbf{U}_i \tilde{\mathbf{w}}^k \geq 1-\delta, i=1,\dots,p, \\ & \tilde{\mathbf{w}}^H \mathbf{A}_i \tilde{\mathbf{w}} \leq \gamma, i=1,\dots,L \\ & \tilde{\mathbf{w}}_i^2 + \tilde{\mathbf{w}}_{i+N}^2 \leq \mathbf{t}_i, i=1,\dots,N. \end{aligned}$$

The optimum deterministic array can be obtained from the support of the K largest weights $\mathbf{w}_i = \tilde{\mathbf{w}}_i + j\tilde{\mathbf{w}}_{i+N}, i=1,\dots,N$.

V. Simulations

In this section, simulation results are presented to validate the proposed antenna selection methods. Consider $K = 8$ available antennas and $N = 16$ uniformly spaced positions with inter-element spacing of $d = \lambda/2$. There are three uncorrelated source signals impinging on the array from directions $\theta_1 = 65^\circ, \theta_2 = 75^\circ, \theta_3 = 110^\circ$ with SNR being 6dB, 3dB and 0dB, respectively. In order to validate the important role of array configurations in determining the adaptive processing performance, that is, the output SNR and beampattern response, we enumerate all 12870 different sparse arrays. Basically, we implement the beamformer in Eq. (4) for each array to calculate the output SNR and peak sidelobe level (PSL), which are presented in Figs. 1 and 2 sorted in an ascending order according to the output SNR. We can make the following observations: (1) For the same beamformer and number of antennas, different array configurations yield different output SNR and beampattern response; (2) The SNR difference between the optimum and worst arrays is 1.45dB. The beampattern of the sparse array with the maximum output SNR exhibits as high as -2.5dB sidelobe level. In terms of the beampattern response, the worst sparse array generates grating lobes, as indicated by the circle in Fig. 2, whereas the best sparse array, shown by square in Fig. 2, can achieve as low as -8.4dB PSL. Combining the two figures, Figs. 1 and 2, it is clear that the optimum sparse array design for quiescent beamforming should put the two metrics, output SNR and beampattern response, into consideration.

We then calculate the optimum 8-antenna sparse array adaptive beamformers in terms of the maximum output SNR with and without controlled sidelobes, utilizing the proposed ILFP algorithm. Note that the ILFP algorithm can obtain the true optimum sparse array without considering the sidelobe constraints as well as the array of a well-shaped quiescent pattern with an acceptable SNR degradation. The two sparse arrays are referred to array (a) and (b) respectively, whereas the sparse array with the lowest PSL is termed array (c). The configurations of the three arrays are depicted in Fig. 3. The respective beampatterns are depicted in Fig. 4. The PSL and output SNRs are listed in the first three rows of Table I. Clearly, array (a) exhibits the highest sidelobes and undesirably low gain towards the third weak source. The sparse array (c) demonstrates the best-shaped beampattern with 15.8dB output SNR. Array (b) circumvents the high

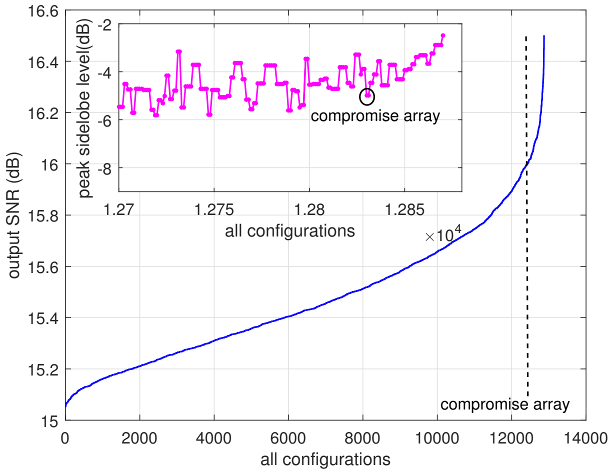

Fig. 1. The output SNR of all 8-antenna sparse arrays sorted in an ascending order. The inner plot shows some arrays' respective PSL.

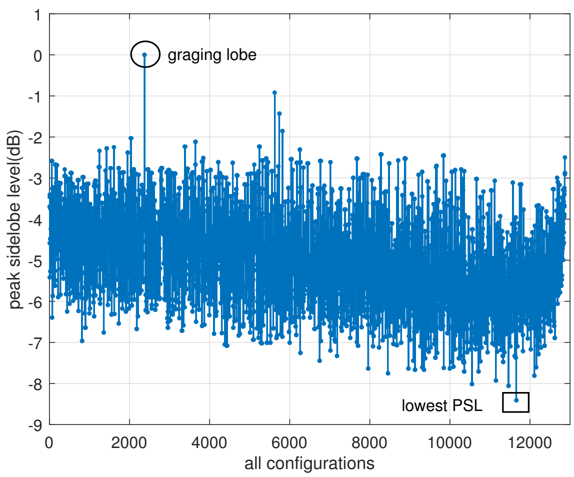

Fig. 2. Peak sidelobe level of all 8-antenna sparse arrays sorted in an ascending order corresponding to the output SNR.

sidelobes in the angular region around the three sources and improves the output SNR to 16dB as well. Thus, array (b) is a reasonable compromise between arrays (a) and (c). Note that the beamforming weights for the three sparse arrays are calculated according to Eq. (4). This validates that array configurations impact the beamforming performance dramatically.

Next, we implement antenna selection to obtain optimum sparse array equal-gain beamformers with and without controlled sidelobes using the same simulation scenario as above. The two selected sparse arrays, namely array (d) and (e), are plotted in Fig. 5. We also depict their respective beampatterns in Fig. 6 using the weight vector $\mathbf{w} = \mathbf{U}\beta$ with $\beta = [1, 1, 1]^T$. The PSL and output SNRs of the two arrays are summarized in the fourth and fifth rows of Table I. We can observe that array configurations also significantly affect the performance of the equal-gain beamformer. By comparing Figs. 4 and 6, it is clear that the beamformer in Eq. (4) ignores the weak source in order to maximize the output SNR. This disadvantage is overcome by the equal-gain beamformer in Eq. (12), as demonstrated in Fig. 6. However, the sacrifice made is the output SNR degradation, which are 15.5dB and 15.2dB for

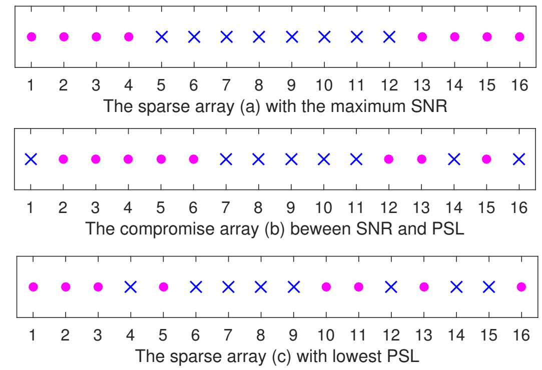

Fig. 3. Optimum 8-antenna sparse arrays in terms of the maximum output SNR beamformer: array (a) without sidelobe constraints, array (b) with controlled sidelobes, array (c) with lowest PSL. The filled dots imply placed antennas and cross the discarded positions.

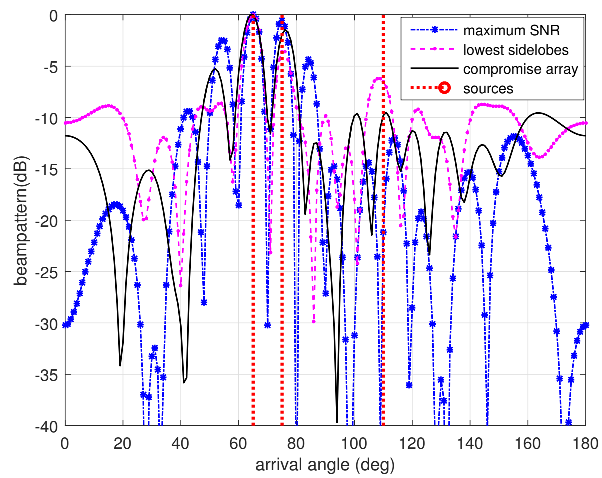

Fig. 4. Beampatterns of three sparse arrays (a), (b) and (c).

the sparse array (d) and (e), respectively. Moreover, array (e) exhibits -6.6dB PSL, compared with -2.9dB of array (d), manifesting a good compromise.

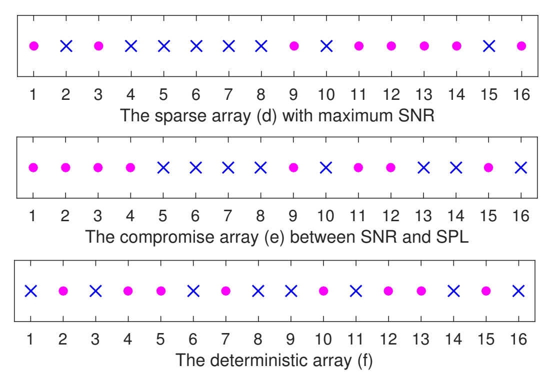

Fig. 5. Selected optimum adaptive and deterministic beamformers with equal sensitivity towards each source: array (c) without sidelobe constraints, array (d) with controlled sidelobes; array (e) deterministic 12-antenna array.

In order to compare the adaptive beamformer design with deterministic array synthesis, we implement the array thinning

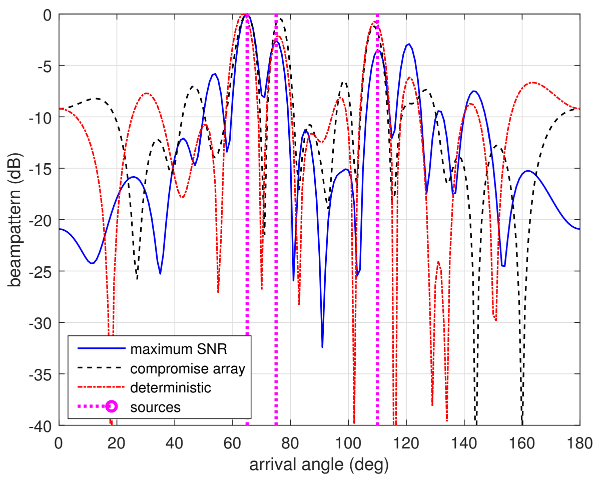

Fig. 6. Beampatterns of two selected adaptive equal-gain beamformers (c) and (d), the green arrow pointing towards undesired high sidelobes.

TABLE I
THE OUTPUT SNR AND PSL OF EACH ARRAY.

Array Name	SNR (dB)	PSL (dB)
Array (a)	16.5	-2.5
Array (b)	16	-5.3
Array (c)	15.8	-8.73
Array (d)	15.5	-2.9
Array (e)	15.2	-6.6
Array (f)	14.87	-6.1

according to Eq. (32). The deterministic array (f) is shown at the bottom of Fig. 5 and the synthesized beampattern is plotted in red dash-dot curve in Fig. 6, with the weights obtained from Eq. (32). The PSL and output SNR of the deterministic array (f) are presented in the last row of Table I. Clearly, the adaptive array beamformer is superior than the deterministic one, which yields the lowest output SNR, that is 14.87dB, although it exhibits well-controlled sidelobe level of -6.1dB, still higher than that of the array (e).

Finally, we summarize the output SNR and PSL of the six sparse arrays (a)-(f) in Table I. No doubt that the array (a) achieves the maximum output SNR, however, it fails to treat all the sources in the field of view with equal sensitivities and exhibits poor quiescent pattern as well. Array (c) presents the best quiescent pattern response with a 0.7dB output SNR degradation. Array (b) manifests a good compromise between the output SNR and the beampattern in terms of the maximum SNR adaptive beamformer. Array (d) is the optimum sparse array with the maximum output SNR under the constraint of equal sensitivities towards all the sources. Array (e) overcomes the disadvantage of high sidelobes of array (d) and a good compromise among arrays (a)-(d). The deterministic array (f) demonstrates the worst output SNR, whereas its advantage is evidently manifested by well-controlled sidelobe levels.

VI. CONCLUSION

We examined the problem of optimum antenna selections in the case of multiple desired sources in interference-free environment with and without applying quiescent pattern constraints. Although superior to deterministic array synthesis, uncontrolled quiescent pattern raises concern when deploying the array with possibility of interference suddenly emerging. We utilized all available degrees of freedom and flexibility in sensor placements to achieve high SNR and desired beampattern. This was accomplished in both cases of maximum output SNR and equal gain beamformers. Simulation results validated the importance of array configurations in determining the beamforming performance in interference-free scenarios.

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