Optimum Sparse Array Receive Beamforming for Wideband Signal Model

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***Abstract*—Sparse array design can potentially achieve comparable performance over uniform array counterparts with a fewer sensors. In this paper, we develop a sparse arrays design method achieving maximum signal-to-interference plus noise ratio (MaxSINR) for wideband source operating in a wideband jamming environment. The problem is formulated as quadratically constraint quadratic program (QCQP) that permits the use of weighted mixed $l_{1,\infty}$-norm squared penalization of the beamformer weight vector to achieve sparse array configurations. We propose the principal eigenvector based technique to control the desired group sparsity while promoting unit rank solutions iteratively. It is shown that the optimum sparse array utilizes the array aperture effectively and provides considerable performance improvement over commonly used arrays. Simulation results are presented to show the effectiveness of proposed algorithm for array configurability under wideband signal model.**

I. Introduction

Sparse array design yields multitude of benefits including high resolution and their ability to cater higher number of sources in the field of view (FOV). Sparse arrays can effectively reduce the computational and hardware overhead of the system while optimizing sensor locations to achieve optimality depending on the signal processing task at hand. Many different metrics like minimum redundancy criteria, extended co-arrays and side-lobe level control have been proposed for optimal sparse array design. These performance metrics results in array designs which are in principle blind to the operating environment and result in efficient structured array topologies [1]–[4]. More recently, the enabling switched antenna and beam technologies have motivated the design for environment adaptive sparse arrays. Maximum signal to noise ratio (MaxSNR) and MaxSINR have been shown to yield significantly efficient beamforming with its performance depending largely on the positions of the sensors as well as the locations of sources in the FOV [5]–[8].

Enhancing the signal power for the desired source operating in an interference active environment is advantageous in many key applications in radar signal processing and medical imaging [9], [10]. The objective of the beamformer is to minimize noise and interference signals at the array output while simultaneously maintaining a desired response in the direction of interest. Capon method is a well known linear constraint beamforming approach that rejects the interference and maximizes the desired source signal power by improving the output signal-to-interference plus noise ratio (SINR) [11].

In this paper, we examine MaxSINR sparse arrays for frequency spread point source operating in wideband interference environment. The wideband sources are common to many applications in array signal processing and are often filtered by employing tapped delay line with sensor array [12]–[14]. Sparse array design for wideband signals have been studied in the context of frequency invariant beampattern design. The end tapered thinned arrays in which the antenna element spacing increases at the array periphery have been shown to have fairly robust beampattern at high frequency ratios [15], [16]. The problem of sparse array MaxSINR has been recently investigated for general rank signal correlation matrices assuming narrowband sources [17], [18].

In order to deal with wideband signal emitters, we pose the problem as optimally selecting K antennas out of N possible equally spaced locations. Each antenna has associated L tapped delay line to jointly process the signal in temporal and spatial domains. Our approach is the natural extension of capon beamforming at the receiver and amounts to maximizing the SINR over all possible sparse array configurations. The antenna selection problem for maximizing SINR amounts to maximizing the principal eigenvalue of the product of the inverse of received data correlation matrix and the desired source correlation matrix [19]. It is an NP hard optimization problem. In order to realize convex relaxation and avoid computational burden of singular value decomposition (SVD) for each possible configuration, we pose this problem as QCQP with weighted $l_{1,\infty}$-norm squared to promote group sparsity. We adopt an iteration based approach to control the sparsity of the optimum weight vector so that K antenna sensors are finally selected. The weighted l_1-norm convex relaxation has been exploited for antenna selection problem for beampattern synthesis, whereas, weighted $l_{1,\infty}$-norm squared relaxation is shown to be very effective for minimizing the required antennas in multicast transmit beamforming [6], [20], [21]. We demonstrate the offerings of the proposed sparse array design by comparing its performance with those of commonly used compact ULA and sparse arrays developed by other design methods.

The rest of the paper is organized as follows: In the next section, we state the problem formulation for maximizing the output SINR under broadband source signal model. Section

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III deals with the optimum sparse array design by semidefinite relaxation and propose iterative algorithm of finding optimum K antenna sparse array design. Simulation and conclusion follow at the end.

II. Problem Formulation

Consider a single desired source and Q interfering source signals impinging on a linear array with N uniformly placed antennas. The baseband received signal $\mathbf{x}(n) \in \mathbb{C}^N$, sampled at the array at time instant n is given by:

$$\mathbf{x}(n) = \mathbf{s}(n) + \sum_{k=1}^{Q} \mathbf{i}_k(n) + \mathbf{v}(n), \quad (1)$$

where $\mathbf{s}(n)$ is the contribution from the desired signal located at θ_l, $\mathbf{i}_k(n)$ are the interfering signal vectors corresponding to the respective directions of arrival, θ_k and $\mathbf{v}(n)$ is the spatially uncorrelated sensor array output noise.

We assume an L tap delay line associated with each antenna sensor. Therefore, we define a stacked vector $\mathbf{X} = [\mathbf{x}^T(n), \mathbf{x}^T(n-1), ..., \mathbf{x}^T(n-L)]^T \in \mathbb{C}^{NL}$ containing the array data collected over L sampling instances. The received signal $\mathbf{X}$ is then combined linearly to maximize the output SINR. The output signal $y(n)$ of the optimum beamformer for maximum SINR is given by [19]:

$$y(n) = \mathbf{w_0}^H \mathbf{X}, \quad (2)$$

where $\mathbf{w_0}$ is obtained by solving the following optimization problem:

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R_{in+n}} \mathbf{w}, \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R_s} \mathbf{w} = 1. \end{aligned} \quad (3)$$

Here, $\mathbf{R_s} = E(\mathbf{SS^H}) \in \mathbb{C}^{NL*NL}$ is the desired signal correlation matrix for $\mathbf{S} = [\mathbf{s}^T(n), \mathbf{s}^T(n-1), ..., \mathbf{s}^T(n-L)]^T$. Likewise, $\mathbf{R_{in+n}}$ is the correlation matrix associated with interference and noise stacked vectors. In the case of frequency spread or wideband source signal, the correlation matrix is given by [22],

$$\mathbf{R_s} = \int_B \int_\Omega \mathbf{S}_\theta(\omega) a(\theta, \omega) a^H(\theta, \omega) d\theta d\omega, \quad (4)$$

In the above equation, Ω and B are the spatial and spectral support of the source signal respectively. For point sources with no significant spatial extent, Eq. (4) becomes,

$$\mathbf{R_s} = \int_B \mathbf{S}_\theta(\omega) a(\theta, \omega) a^H(\theta, \omega) d\omega, \quad (5)$$

The space time steering vector $\mathbf{a}(\theta, \omega)$ corresponding to the source signal can be represented as a Kronecker product:

$$\mathbf{a}(\theta, \omega) = \boldsymbol{\phi}_\omega \otimes \mathbf{a}_\theta(w), \quad (6)$$

with,

$$\boldsymbol{\phi}_\omega = [1 \;\; e^{j(\pi\omega/\omega_{max})} \dots e^{j(\pi\omega/\omega_{max})(L-1)}]^T. \quad (7)$$

$$\begin{aligned} \mathbf{a}_\theta(w) &= [1 \;\; e^{j(2\pi/\lambda)dcos(\theta_k)} \dots e^{j(2\pi/\lambda)d(N-1)cos(\theta_k)}]^T, \\ &= [1 \;\; e^{j(\pi\omega/\omega_{max})cos(\theta_k)} \dots e^{j(\pi\omega/\omega_{max})(N-1)cos(\theta_k)}]^T. \end{aligned} \quad (8)$$

Here ω_{max} corresponds to the maximum allowable frequency as to avoid aliasing in the temporal domain. Similarly, we set the inter-element spacing $d = \lambda_{min}/2$ to avoid spatial aliasing corresponding to ω_{max}, where λ is the wavelength associated with ω. The correlation matrix for broadband interferers $\mathbf{i}_k$ is defined according to Eq. (5) with the respective θ_k and B_k. The sensor noise correlation matrix, $\mathbf{R_n} = \sigma_v^2 \mathbf{I}$ assuming spatially and temporally uncorrelated noise $\mathbf{v}(n)$ with variance σ_v^2. The constraint minimization problem in Eq. (3) can be written equivalently by replacing $\mathbf{R_{in+n}}$ with $\mathbf{R} = \mathbf{R_s} + \mathbf{R_{in+n}}$ as follows [19],

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R} \mathbf{w}, \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R_s} \mathbf{w} = 1. \end{aligned} \quad (9)$$

The analytical solution of the above optimization problem exists and is given by $\mathbf{w_0} = \mathscr{P}\{\mathbf{R_{in+n}}^{-1}\mathbf{R_s}\} = \mathscr{P}\{\mathbf{R}^{-1}\mathbf{R_s}\}$. The operator $\mathscr{P}\{.\}$ computes the principal eigenvector of it's argument. The corresponding optimum output $SINR_o$:

$$SINR_o = \frac{\mathbf{w_0^H R_s w_0}}{\mathbf{w_0^H R_{in+n} w_0}} = \Lambda_{max}\{\mathbf{R_{in+n}^{-1} R_s}\}. \quad (10)$$

Equation (10) shows that the optimum beamformer for maximizing SINR is directly related to the desired and interference plus noise correlation matrix.

III. Optimum sparse array design

The problem of maximizing the principal eigenvalue of the correlation matrices associated with K antenna selection is a combinatorial optimization problem. To proceed, we assume that the antenna configuration remains the same within the observation time L, and, therefore we need to ensure that same K antennas are selected at each sampling instance within the coherent interval L. This is achieved by optimally selecting K entries from the first N elements of $\mathbf{w}$ and the same K entries from each subsequent block of N elements in $\mathbf{w}$. There are L such blocks. We assume that either the full array data correlation matrix is known or has a co-array that delivers the correlation values across all array elements [23]. Define $\mathbf{w}_k \in \mathbb{C}^L$ to be the weights corresponding to tap delay line of kth sensor. Then, the problem formulated in (5) can be rewritten as follows:

$$\begin{aligned} \underset{\mathbf{w} \in \mathbb{C}^{\mathbf{NL}}}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R} \mathbf{w} + \mu(\sum_{k=1}^{N} ||\mathbf{w}_k||_p), \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R_s} \mathbf{w} = 1. \end{aligned} \quad (11)$$

Here, $||.||_p$ denotes the p-norm of the vector. The mixed $l_{1,p}$-norm regularization is known to thrive the group sparsity in the solution for $p > 1$ as is required in our case. The relaxed problem expressed in Eq. (11) induces the group sparsity in optimal weight vector $\mathbf{w}$ without placing a hard constraint on the specific cardinality of $\mathbf{w}$. The problem (11) can be penalized instead by the weighted l_1-norm function which is a well known sparsity promoting formulation [24],

$$\begin{aligned} \underset{\mathbf{w} \in \mathbb{C}^{\mathbf{NL}}}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R} \mathbf{w} + \mu(\sum_{k=1}^{N} u^i(k) ||\mathbf{w}_k||_p), \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R_s} \mathbf{w} = 1. \end{aligned} \quad (12)$$

where, $\mathbf{u^i}$ is the sparsity promoting weight vector at the ith iteration. The ∞-norm is choosen for the p-norm and weighted

TABLE I: Algorithm to achieve desired cardinality of optimal weight vector $\mathbf{w_0}$.

	Steps of proposed algorithm
Step 1	Initialize the weight matrix $\mathbf{U^i}$ to all ones and appropriate small values of μ and ϵ.
Step 2	Run the rank relaxed SDP of Eq. (15). Check if some entries in $\tilde{\mathbf{W}}$ is exactly zero, if yes, check the cardinality of non zero column of $\tilde{\mathbf{W}}$ and go to Step 1 and increase or decrease the value of μ to enhance or reduce the sparsity respectively until desired cardinality is achieved. If desired cardinality is achieved go to Step 4 otherwise, in case of no-non zero values go to Step 3.
Step 3	Approximate solution matrix by the principal eigenvector and subsequently update the weight vector $\mathbf{U^i}$ according to Eq. (16) and repeat Step 2.
Step 4	After achieving the desired cardinality, run SDR for reduced size correlation matrix corresponding to nonzero values of $\tilde{\mathbf{W}}$ and $\mu = 0$, yielding, $\mathbf{w_0} = \mathscr{P}\{\mathbf{W}\}$.

l_1-norm function in (12) is replaced by the l_1-norm squared function without effecting the regularization property of the weighted l_1-norm function [6] i.e.,

$$\begin{aligned} \underset{\mathbf{w} \in \mathbb{C}^{\mathbf{NL}}}{\text{minimize}} \quad & \mathbf{w}^H \mathbf{R} \mathbf{w} + \mu \Big(\sum_{k=1}^{N} u^i(k) \|\mathbf{w}_k\|_\infty\Big)^2, \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{R_s} \mathbf{w} = 1. \end{aligned} \tag{13}$$

The semidefinite program (SDP) of the above problem can then be realized by replacing $\mathbf{W} = \mathbf{w}\mathbf{w^H}$. Re-expressing the quadratic function, $\mathbf{w}^H\mathbf{R}\mathbf{w} = \text{Tr}(\mathbf{w}^H\mathbf{R}\mathbf{w}) = \text{Tr}(\mathbf{R}\mathbf{w}\mathbf{w}^H) = \text{Tr}(\mathbf{R}\mathbf{W})$, where Tr(.) is the trace of the matrix. This expression yields the following problem [6], [25], [26],

$$\begin{aligned} \underset{\mathbf{W} \in \mathbb{C}^{\mathbf{NL*NL}}, \tilde{\mathbf{W}} \in \mathbb{R}^{\mathbf{N*N}}}{\text{minimize}} \quad & \text{Tr}(\mathbf{R}\mathbf{W}) + \mu \text{Tr}(\mathbf{U^i}\tilde{\mathbf{W}}), \\ \text{s.t.} \quad & \text{Tr}(\mathbf{R_s}\mathbf{W}) = 1, \\ & \tilde{\mathbf{W}} \geq |\mathbf{W}_{dd}| \quad \forall d \in \ 1, 2, ...L., \\ & \mathbf{W} \succeq 0, \ \text{Rank}(\mathbf{W}) = 1. \end{aligned} \tag{14}$$

Here, $\mathbf{W}_{dd} \in \mathbb{C}^{N*N}$ is the dth diagonal block matrix of $\mathbf{W}$. The rank constraint in Eq. (14) is non convex and, therefore, we drop the rank constraint resulting in the following semidefinite relaxation (SDR):

$$\begin{aligned} \underset{\mathbf{W} \in \mathbb{C}^{\mathbf{NL*NL}}, \tilde{\mathbf{W}} \in \mathbb{R}^{\mathbf{N*N}}}{\text{minimize}} \quad & \text{Tr}(\mathbf{R}\mathbf{W}) + \mu \text{Tr}(\mathbf{U^i}\tilde{\mathbf{W}}), \\ \text{s.t.} \quad & \text{Tr}(\mathbf{R_s}\mathbf{W}) \geq 1, \\ & \tilde{\mathbf{W}} \geq |\mathbf{W_{dd}}| \quad \forall d \in \ 1, 2, ...L., \\ & \mathbf{W} \succeq 0. \end{aligned} \tag{15}$$

A. Unit rank promoting iteration

As suggested in [24], the weight matrix $\mathbf{U^i}$ is initialized unweighted i.e. by all ones matrix and iteratively updated as follows,

$$\mathbf{U}_{m,n}^{i+1} = \frac{1}{|\mathbf{W}_{m,n}^{i}| + \epsilon}. \tag{16}$$

However, for the underlying problem, the solution matrix $\mathbf{W}$ is not exactly rank one matrix at each iteration. Therefore, the weight matrix iteratively favors solution of higher ranks. To mitigate this problem, we approximate the solution matrix by rank 1 matrix, $\mathbf{W}^i = \mathscr{P}\{\mathbf{W}^i\}\mathscr{P}^H\{\mathbf{W}^i\}$, to promote unit rank solutions iteratively. This approach is found to be very effective for the selection of optimum antenna sensors. The proposed algorithm for controlling the sparsity of the optimal weight vector $\mathbf{w_0}$ is summarized in TABLE. I.

IV. Simulations

In this section, we show the effectiveness of our proposed technique for the sparse array design for wideband sources based on MaxSINR criterion. The importance of array configurability for MaxSINR is further emphasized and reinforced by comparing the optimum sparse array design with various array configurations, under same signal model. For all our examples, we select $K = 8$ sensors from $N = 14$ possible equally spaced locations with inter-element spacing of $\lambda_{min}/2$. The array data is sampled periodically at sampling frequency of 1 Hz. For our example, we have 6 delay line filter taps available with each selected antenna sensor.

A frequency spread desired point source is impinging on a linear array from DOA 65^0. The normalized frequency spread of the source is from 0.4 Hz to 0.5 Hz. Four strong wideband interferers are operating from 40^0, 50^0, 75^0 and 150^0. All of these interferers are occupying the full frequency band from 0 Hz to 0.5 Hz. The SNR of the desired signal is 0 dB, and the INR of each interfering signals is set to 30 dB. Figure 1 shows the frequency dependent beampattern for the optimum array configuration recovered through SDR. It is evident from the beampattern that high gain is maintained throughout the band of interest for the desired source signal while mitigating the interferers for all possible frequencies. It is important to note that the proposed algorithm performs very close to the optimum array found by exhaustive search (3003 possible configurations), which has very high computational cost attributed to expensive singular value decomposition (SVD) for each enumeration. The maximum possible SINR for the optimum array configuration found through exhaustive search comes to be 10.46 dB, whereas the array configuration found through convex relaxation is 10.11 dB which is only around 0.3 dB down as compared to the optimum array found through enumeration. The case of this MaxSINR arises when the interferers are considerably alleviated jointly by the array topology along with the optimum weight vectors. For the relaxed SDP, we initialize small values for μ and ϵ (10^{-2} and 10^{-5} respectively in our case). On average, the proposed algorithm takes six to seven iterations to converge at the optimum locations and number of sensors; hence, offering dramatic saving in the computational cost. It is of interest to compare the optimum sparse array performance with the compact ULA. The output SINR for the compact ULA with 8 antennas is 5 dB which is more than 5 dB down from the optimum sparse array design. It is evident from Fig. 2 that this performance degradation of the compact ULA is attributed to lesser gain for all frequencies of interest of desired source direction while mitigating interferers over all possible frequencies.

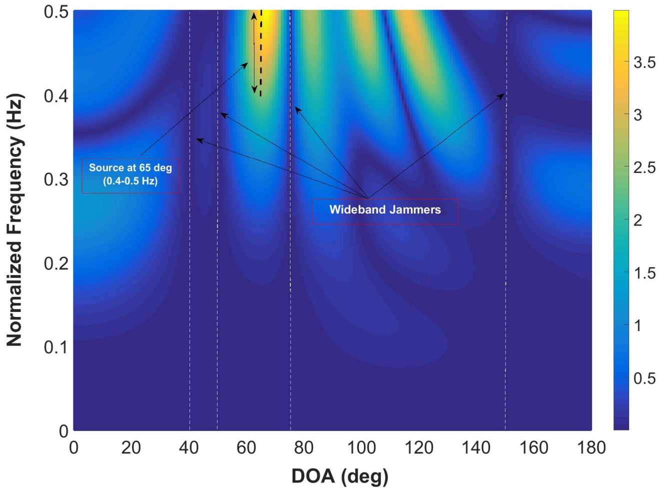

Fig. 1: Frequency dependent beampattern for the optimum array recovered through enumeration.

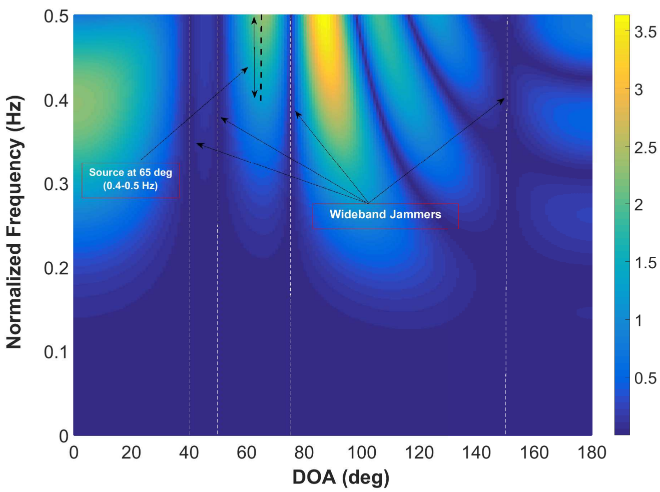

Fig. 2: Frequency dependent beampattern for the optimum array recovered through convex relaxation.

We also consider the case of optimizing the sparse array design for the above mentioned scenario while ignoring the temporal filter coefficients and only optimizing for the spatial filter. To maintain fair comparison, the array optimized over spatial degrees of freedom is subsequently applied to the delay line scenario. The beampattern is shown in the Fig. 3. The output SINR obtained is 4.62 dB that is more deteriorated than the performance of compact ULA, and performance degradation is evident from the beampattern. This degradation is attributed to the disjoint design of the optimum spatial and temporal filters. Finally, we plot the beampattern for the array configuration which results in the worst SINR performance. The worst case beampattern is shown in the Fig. 4, and offers minimum gain towards the direction and frequencies of interest while ensuring interference mitigation. The corresponding

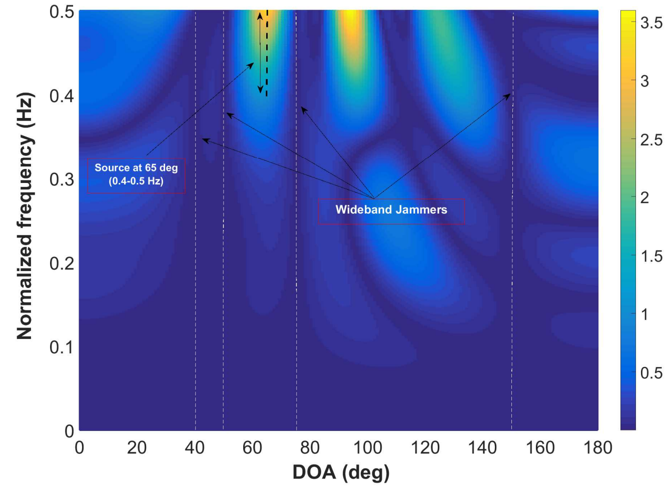

Fig. 3: Frequency dependent beampattern for the optimum array ignoring tap delay line.

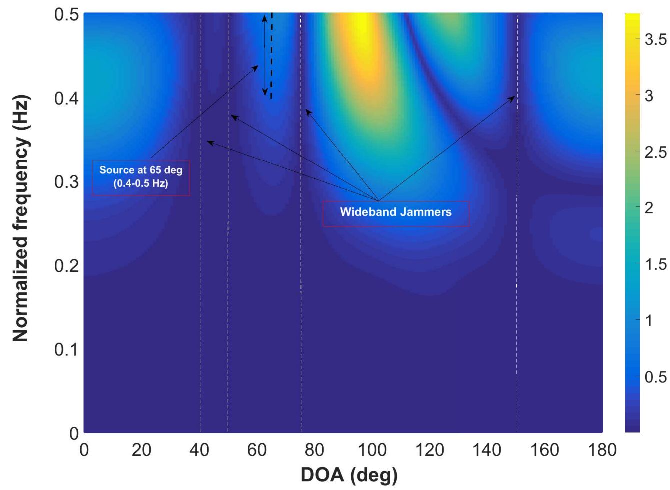

Fig. 4: Frequency dependent beampattern for the array configuration with the worst performance.

output SINR in this case is -3.86 dB which is considerably less than that of the optimum design and other array topologies discussed above.

Figure 5 shows various sparse array topologies for the cases under consideration (where ”.” and ”$\times$” represent the presence and absence of sensor respectively). The optimum sparse array maximizing the SINR and obtained through exhaustive search is shown in the Fig. 5a, whereas the optimum sparse array recovered through SDR is shown in the Fig. 5b. Both of these arrays occupy similar aperture and have comparable performances. The sparse array ignoring the temporal filter and the array with the worst performance for the given scenario are shown in the Fig. 5c and Fig. 5d, respectively. It is of interest to point out that both these arrays utilize greater aperture than the optimum array configuration, yet they perform so poorly.

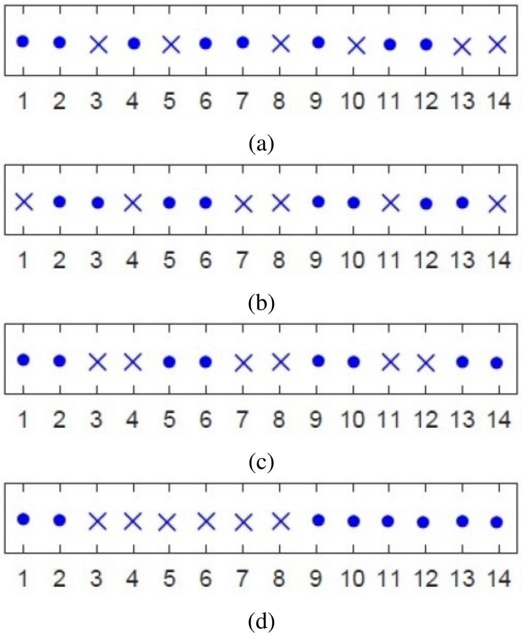

Fig. 5: (a) Optimum 8 antenna array (enumeration) (b) Optimum 8 antenna array (SDR) (c) 8 antenna array ignoring tap delay line (d) Worst performance 8 antenna array

This underscores the importance of sparse array design under the metric of maximizing the SINR which does not simply lends itself to favoring large aperture arrays.

V. Conclusion

This paper considered optimum sparse array configuration for maximizing the beamformer output SINR for the case of broadband signal models. It was shown that the weighted mixed $l_{1,\infty}$-norm squared group sparsity promoting penalization with principal eigenvector based iterative sparsity control algorithm is particularly effective in finding the optimum sparse array design with low computational complexity. We showed the effectiveness of our approach for the frequency spread source operating in wideband jamming environment. The MaxSINR optimum sparse array yielded considerable performance improvement over compact ULA and other sparse arrays for the underlying scenario. We solved the optimization problem by both the proposed algorithm and enumeration and showed strong agreement between the two methods in terms of array performance. The proposed approach can be easily extended to joint optimization in both spatial and temporal domains.

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